CHAPTER 6

MATHEMATICS

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Section

For ease in reference

Mathematical signs	see Chapter 38 Sect. 6(vii)
Greek letters	see Chapter 38 Sect. 14
Definitions	see Chapter 38 Sect. 15
Decimal equivalents of fractions	see Chapter 38 Sect. 16
Numerical values	see Chapter 38 Sect. 18
Logarithm tables	see Chapter 38 Sect. 20
Trigonometrical and hyperbolic tables	see Chapter 38 Sect. 21
Log. scales and log. scale interpolator	see Chapter 38 Sect. 22

Mathematics, to the radio engineer, is merely a tool to be used in his design work. For this reason it is often used in a slovenly manner or with insufficient precision or understanding.

There are normally three stages in the solution of a problem—

1. Transferring the mechanical or electrical conditions into a mathematical form.
2. Solving the mathematics.
3. Interpreting and applying the mathematical solution.

The first stage is dealt with in Chapter 4 ; the second stage is the subject of this chapter, while the third stage requires careful consideration of all the relevant conditions. A solution only applies under the conditions assumed in stage one, which may involve some approximations and limits. In all cases the solution should be checked either experimentally or theoretically to prove that it is a true solution.

This chapter is not a textbook on mathematics, although it is in such an easy form that anyone with the minimum of mathematical knowledge should be able to follow it. It has been written primarily for those who require assistance in " brushing up " their knowledge, and for the clarification of points which may be imperfectly understood. It is " basic " rather than elementary in its introduction, and could therefore be read with advantage by all.

Sufficient ground is covered for all normal usage in radio receiver design, except for that required by specialists in network and filter design.

Reference data have been included for use by all grades

SECTION 1 : ARITHMETIC AND THE SLIDE RULE

(i) Figures (ii) Powers and roots (iii) Logarithms (iv) The slide rule (v) Short cuts in arithmetic

(i) Figures

A figure (e.g. 5) indicates a certain number of a particular object—e.g. five resistors or five radio receivers ; or else five units of a particular scale—e.g. five inches, five microfarads or five ohms. No matter what we may do with adding, subtracting, multiplying or dividing, if we begin with ohms we must finish with ohms, and so on with any other unit. We cannot add together dissimilar objects without identifying each type, and similarly with any other mathematical process.

(ii) Powers and roots

As figures are often too large, or too small, to be shown completely in the ordinary form, there is a Scientific Notation commonly used—

Numbers above unity	Numbers below unity
$10^1 = 10$	$10^{-1} = 0.1$
$10^2 = 100$	$10^{-2} = 0.01$
$10^3 = 1\,000$	$10^{-3} = 0.001$
$10^4 = 10\,000$	$10^{-4} = 0.000\,1$
$10^5 = 100\,000$	$10^{-5} = 0.000\,01$
$10^6 = 1\,000\,000$	$10^{-6} = 0.000\,001$
	and so on

$$10^0 = 1$$

Note that 10^2 means $10 \times 10 = 100$; 10^3 means $10 \times 10 \times 10 = 1000$ and so on. The " 2 " and the " 3 " are called exponents, or indices (plural of index).

10^2 is called " 10 squared "
10^3 is called " 10 cubed "
10^4 is called " 10 to the fourth " etc.
10^{-1} is called " 10 to the minus 1 " etc.

Examples : 2 750 000 is written 2.75×10^6
0.000 025 is written 2.5×10^{-5}.

Multiplying is carried out as in the examples :

$$(1.5 \times 10^6) \times (4 \times 10^{-3}) = 6 \times 10^3$$
$$(4.5 \times 10^3) \times (2 \times 10^2) = 9 \times 10^5$$

Reference should be made to the table of multiples and sub-multiples in Chapter 38 Sect. 17, Table 59.

The same procedure may be applied, not only to 10, but to any figure, e.g. 3 ;

$3^0 = 1$
$3^1 = 3$
$3^2 = 3 \times 3 = 9$ $\quad$ $3^{-2} = 1/3^2 = 1/9$
$3^3 = 3 \times 3 \times 3 = 27$ $\quad$ $3^{-3} = 1/3^3 = 1/27$

Roots : The expression $9^{\frac{1}{2}}$ may be written $\sqrt{9}$, where the sign $\sqrt{\ }$ is called the square root. Similarly $27^{1/3}$ may be written $\sqrt[3]{27}$ where the sign $\sqrt[3]{\ }$ is called the cube root.

The whole question is dealt with more fully, and more generally under " Algebra " in Sect. 2.

(iii) Logarithms

We may write $100 = 10^2$
or we may express this in different language as

$$2 = \log_{10} 100$$

which is spoken of as " log 100 to the base 10." Here 2 is the logarithm of 100 to the base 10.

Tables of logarithms to the base 10 are given in Chapter 38 Sect. 20, Table 71. They are useful in multiplying and dividing numbers which are too large to handle conveniently by the ordinary procedure.

A typical logarithm is 2.4785. Here the figure 2 to the left of the decimal point is called the **index**, and the figures 4785 to the right of the decimal point are called **the mantissa**.

(1) The **index** of the logarithm of a number greater than unity is the number which is less by one than the number of digits (figures) in the integral* part of the given number; for example the index of the logarithm of

57 640	is 4
5 764	is 3
576.4	is 2
57.64	is 1
5.764	is 0

If the number is less than unity, the index is negative, and is a higher number by one than the number of zeros that follow the decimal point of the given number; for example the index of the logarithm of

.576 4	is −1
.005 764	is −3.

To denote that the index only is negative, the minus sign is usually written above it;

e.g. $\bar{1}$, $\bar{3}$.

(2) The **mantissa** of the logarithm is found from the tables. Proceed to find the first two figures in the left hand column of the table, then pass along the horizontal line to the vertical column headed by the third figure. To this number add the number in the difference column under the fourth figure of the given number. The mantissa is the result obtained by this process with a decimal point before it.

For example, to find $\log_{10} 5764$.

The index is 3

The mantissa is .7604 + .0003 = .7607

Therefore $\log_{10} 5764 = 3.7607$.

Similarly, $\log_{10} 0.5764 = \bar{1}.7607$ (note that the index is negative, but the mantissa positive).

To find the number whose logarithm is given, it is possible to use either antilog. tables (if these are available) or to use the log. tables in the reverse manner. In either case, only the mantissa (to the right of the decimal point) should be applied to the tables.

The procedure with log. tables is firstly to find the logarithm, on the principal part of the table, which is next lower than the given logarithm, then to calculate the difference, then to refer to the difference columns to find the number—exactly the reverse of the previous procedure.

For example, to find the number whose logarithm is 2.5712. The mantissa is .5712 and the nearest lower logarithm on the tables is 5705, the difference being 7 (in the fourth figure). The number whose log = 5705 is 3720, and to this must be added the figure 6 which corresponds to the difference of 7 in the fourth column; the number is therefore 3720 + 6 = 3726. The decimal point must be placed so as to give 2 + 1 = 3 digits to the left of the point, i.e. 372.6.

The application of logarithms

If two numbers are to be **multiplied** together, the answer may be found by adding their logarithms, and then finding the number whose logarithm is equal to their sum For example, suppose that it is desired to multiply 371.6 × 58.24,

$$\begin{aligned} \log 371.6 &= 2.5701 \\ \log 58.24 &= 1.7652 \\ \hline \text{sum} &= 4.3353 \end{aligned}$$

The number whose log = .3353 is 2164.

The decimal point should be placed so as to give 4 + 1 = 5 digits to the left of the point; i.e. 21 640.

*To the left of the decimal point.

Therefore $371.6 \times 58.24 = 21\,640$.

It should be remembered that the last figure of four figure logs. is correct to the nearest unit, and that slight errors creep into the calculations through additions and other manipulations. The first three digits of the answer will be exact, and the fourth only approximate.

If one number is to be **divided** by a second number, the answer may be found by subtracting the logarithm of the second from the logarithm of the first, then finding the number whose logarithm is equal to their difference.

Logarithms may also be used to find the **powers of numbers** For example, to find the value of $(3.762)^3$:

$$\log (3.762)^3 = 3 \log 3.762 = 3 \times .5754$$
$$= 1.7262$$

Therefore $(3.762)^3 = \text{antilog } 1.7262 = 53.23$.

For other applications of logarithms see Sect. 2(xvii)

(iv) The Slide Rule

The Slide Rule is a mechanical device to permit the addition and subtraction of logarithms so as to effect multiplication and division of the numbers. The small size of the normal slide rule does not give as high a degree of accuracy as four figure log. tables, but is sufficiently accurate for many calculations.

The usual 10 inch slide rule has four scales A, B, C, D of which B and C are on the slide. The scales C and D (the lower pair) are normally used for multiplication and division, and each covers from 1 to 10. The upper scales A and B cover from 1 to 100. The **square** of any number from 1 to 10 is found by adjusting the line on the cursor (runner) to fall on the number on the D scale, and reading the answer where the line cuts the A scale. The **cube** of a number from 1 to 10 may be found by squaring, and then multiplying the result by the original number on the B scale, reading off the answer on the A scale. If the number is not between 1 and 10, firstly break it up into factors, one of which should be a multiple of 10, and the other a number between 1 and 10, then proceed as before. For example

$$300^2 = (3 \times 100)^2 = (3)^2 \times (100)^2 = 3^2 \times 10^4.$$

The value of 3^2 is found in the normal way to be 9 ; this is then multiplied by 10^4 to give the answer $9 \times 10^4 = 90\,000$.

Square roots may be found by the reverse procedure. Firstly reduce the number to factors, one of which should be a multiple of 100 and the other between 1 and 100, then apply the cursor to the number on the A scale and read the answer on the D scale to be multiplied by the square root of the 100 factor. For example, to find the square root of 1600 : $\sqrt{1600} = \sqrt{16 \times 100} = \sqrt{16} \times \sqrt{100} = \sqrt{16} \times 10$.

The value of 4 on the D scale is then multiplied by 10 to give the answer 40.

Cube roots of numbers between 1 and 100 may be determined by setting the cursor to the number on the A scale, then moving the slide until the B scale cursor reading is the same as the D scale reading below 1 on the C scale.

Slide rules which have log/log scales may be used to determine any power of a number 1.1 or greater (up to a maximum value of 100 000). Set the cursor to the number on the upper log/log scale, then set 1 on the C scale to the same cursor line. Move the cursor to the required power on the C scale and read the answer on the log/log scale. If the number is too high to be on the upper log/log scale, carry out the same procedure on the lower log/log scale. If the number is found on the upper scale, but the answer is beyond the limits of this scale, set the mark* (e.g. W) on the slide immediately below the number on the upper scale, and read the answer on the lower scale, immediately below the power on the C scale.

If several figures are to be multiplied and divided, carry out multiplication and division alternately, e.g.

$$\frac{75 \times 23 \times 5}{41 \times 59 \times 36}$$

should be handled as $75 \div 41 \times 23 \div 59 \times 5 \div 36$.

*With slide rules having no special mark, use 10 on the C scale as the "mark."

In a complicated calculation, especially with very large and very small numbers, it is highly desirable to arrange the numerator and denominator in powers of 10. For example

$$\frac{75\,000 \times 0.0036 \times 5900}{160\,000 \times 0.000\,001\,7} = \frac{7.5 \times 10^4 \times 3.6 \times 10^{-3} \times 5.9 \times 10^3}{1.6 \times 10^5 \times 1.7 \times 10^{-6}}$$
$$= \frac{7.5 \times 3.6 \times 5.9 \times 10^5}{1.6 \times 1.7}$$

The slide rule does not indicate the position of the decimal point, and it is necessary to determine the latter by some method such as inspection; this is much easier when the individual numbers are all between 1 and 10 as in the example above. It is also possible to keep track of the decimal point by noting how often the manipulation passes from end to end of the rule.

To find the logarithm of a given number, move the 1 on C scale to the number on the D scale, then turn the rule over and read the logarithm on the L scale against the mark (this will be a number between 0 and 1).

To find the decibels corresponding to a ratio, proceed as for the logarithm, but multiply by 10 for a power ratio or 20 for a voltage ratio.

To find the sine or tangent of an angle, first set the angle on the S or T scale to the mark, then read the value on the B scale, below 1 on the A scale, and divide by 100.

There are countless special types of slide rules, and in all such cases the detailed instructions provided by the manufacturers should be studied.

Hints on special calculations on the slide rule

(1) $Z = \sqrt{R^2 + X^2} = R\sqrt{1 + (X^2/R^2)}$

Procedure: For example if $X = 3$ and $R = 2$ set cursor to 3 on D scale, move slide to give 2 on C scale. The value of $(X/R)^2$ is given by the value on A scale opposite 1 on B scale—in this case 2.25. Move the slide up to 3.25 (= 2.25 + 1) and then move the cursor to 2 on C scale, reading 3.61 on D scale as the answer.

(2) If a large number of figures is to be divided by one figure, divide unity or 10 (D scale) by the divisor (C scale) and then, with fixed slide, move the cursor to each dividend in turn on the C scale, reading the answer on the D scale.

(v) Short cuts in arithmetic

(a) Approximations involving π

π^2 may be taken as 10 with an error less than 1.4%
π may be taken as 25/8 with an error less than 0.6%
$1/\pi$ may be taken as 8/25 with an error less than 0.6%
2π may be taken as 25/4 with an error less than 0.6%
$1/2\pi$ may be taken as 4/25 with an error less than 0.6%
$(2\pi)^2 = 39.5$ with an error less than 0.06%

(b) Approximations with powers and roots

General relation: $(x \pm \delta)^n \approx x^n \pm nx^{n-1}\delta$

where δ is small compared with x. Examples are given below.

Squares ($n = 2$)

$10.1^2 = (10 + 0.1)^2 \approx 10^2 + 2 \times 10 \times 0.1 \approx 102$ (error = 0.01%)
$10.2^2 = (10 + 0.2)^2 \approx 10^2 + 2 \times 10 \times 0.2 \approx 104$ (error = 0.04%)
$9.9^2 = (10 - 0.1)^2 \approx 10^2 - 2 \times 10 \times 0.1 \approx 98$ (error = 0.01%)

Square roots [$n = 0.5$; $(n - 1) = -0.5$]

$$\sqrt{x \pm \delta} \approx \sqrt{x} \pm \frac{0.5\delta}{\sqrt{x}}$$

$$\sqrt{101} = \sqrt{100 + 1} \approx \sqrt{100} + \frac{0.5 \times 1}{\sqrt{100}} \approx 10.05 \text{ (error} = 0.001\%)$$

$$\sqrt{110} = \sqrt{100 + 10} \approx \sqrt{100} + \frac{0.5 \times 10}{\sqrt{100}} \approx 10.5 \text{ (error} = 0.12\%)$$

$$\sqrt{50} = \sqrt{49 + 1} \approx \sqrt{49} + \frac{0.5 \times 1}{\sqrt{49}} \approx 7.0714 \text{ (error} = 0.004\%)$$

Cubes ($n = 3$)

$(10.2)^3 = (10 + 0.2)^3 \approx 10^3 + 3 \times 10^2 \times 0.2 \approx 1060$ (error $= 0.1\%$)

Cube roots [$n = 0.333$; $(n - 1) = -0.667$]

$$\sqrt[3]{x \pm \delta} \approx \sqrt[3]{x} \pm \frac{\delta}{3(\sqrt[3]{x})^2}$$

$$\sqrt[3]{66} = \sqrt[3]{64 + 2} \approx \sqrt[3]{64} + \frac{2}{3 \times 4^2} \approx 4.042$$

For more accurate approximations see Sect. 2 eqns. (82) and (83).

(c) Approximations in multiplication

$a \times b \approx \frac{1}{4}(a + b)^2$ where a and b are close together

e.g. $49 \times 51 \approx \frac{1}{4}(49 + 51)^2 \approx \frac{1}{4}(100)^2 \approx 2500$ (error 0.04%).

This may be put into the alternative form :

$a \times b \approx$ (arithmetical mean between a and $b)^2$

e.g. $68 \times 72 \approx (70)^2 \approx 4900$ (error 0.08%).

(d) Exact multiplication

$$a \times b = \tfrac{1}{4}\{(a + b)^2 - (a - b)^2\} \text{ (no error)}$$
$$= (\text{arithmetical mean between } a \text{ and } b)^2 - \tfrac{1}{4}(a - b)^2.$$

When $(a - b) = 1$, the second term in this expression becomes $\frac{1}{4}$ and we have the exact application :

$$3\tfrac{1}{2} \times 4\tfrac{1}{2} = 16 - \tfrac{1}{4} = 15\tfrac{3}{4} \text{ (exact).}$$

Another application is illustrated by the example

$$98 \times 102 = 100^2 - \tfrac{1}{4}(4)^2 = 10\,000 - 4 = 9996.$$

(e) To multiply by 11

To multiply a number by 11, write down the last figure, add the last and last but one and write down the result, carrying over any tens to the next operation, add the last but one and the last but two and so on, finishing by writing down the first

e.g. $11 \times 42\,736 = 470096$ (no error)

(f) For approximations based on the Binomial Theorem see Sect. 2(xviii).

(g) For general approximations see Sect. 2(xx).

SECTION 2 : ALGEBRA

(i) Addition (ii) Subtraction (iii) Multiplication (iv) Division (v) Powers (vi) Roots (vii) Brackets and simple manipulations (viii) Factoring (ix) Proportion (x) Variation (xi) Inequalities (xii) Functions (xiii) Equations (xiv) Formulae or laws (xv) Continuity and limits (xvi) Progressions, sequences and series (xvii) Logarithmic and exponential functions (xviii) Infinite series (xix) Hyperbolic functions (xx) General approximations.

See Section 6 for Complex Algebra.

Algebra is really only arithmetic, except that we use alphabetical symbols to stand for figures. It is frequently more convenient to put an expression into an algebraic form for general use, and then to apply it to a particular case by writing figures in place of the letters. All algebraic expressions are capable of being converted into arithmetical ones, and the fundamental mathematical processes of algebra may be used in arithmetic.

(i) Addition

If a, b, and c are all values of the one unit (e.g. all resistances in ohms) we can add them together to find the sum d, where d will also be in the same unit,

$$d = a + b + c$$

For example, if $a = 5$, $b = 10$, $c = 15$ ohms,
then $d = 5 + 10 + 15 = 30$ ohms.

(ii) Subtraction

Subtraction is the opposite of addition, or negative addition, and can only be applied when the quantity to be subtracted is in the same unit as the quantity from which it is to be taken. For example let

$$a = b - c$$

where a, b and c are all voltages.

If $b = 6$ volts and $c = 2$ volts, then

$$a = 6 - 2 = 4 \text{ volts.}$$

As another example, let a, b and c be readings of a thermometer in degrees—say $b = 10°C$ and $c = 20°C$, then $a = 10° - 20° = -10°C$. This is commonly described as " 10 degrees below zero " or " a temperature of minus 10 degrees." Thus a negative temperature has a definite value and is readily understood. Its magnitude is given by the figure, while its direction above or below zero is given by the positive or negative sign.

Similarly a negative current is one with a magnitude as indicated by the figure but with a direction opposite to that of a positive current. In most cases the direction of a positive current is arbitrarily fixed, and if the answer comes out negative it merely indicates the actual direction of current flow is the opposite of the direction assumed. The same principle holds in all cases.

(iii) Multiplication

Multiplication is continued addition—

$$4 \times a = a + a + a + a$$

and is commonly written as $4a$. In the general sense we can write ba where b is any number ; this has the same value as ab,

$$\text{i.e. } ab = ba \text{ or } a.b = b.a.$$

It should be noted that

$$4 \times (-a) = -4a$$
$$\text{and } (-1) \times (-1) = +1.$$

(iv) Division

Division is the breaking up of a number of things into a given number of groups, e.g.

$$6a \div 3 = (2a + 2a + 2a) \div 3 = 2a.$$

We may write this in the alternative forms

$$\frac{6a}{3} = 2a, \text{ or } 6a/3 = 2a.$$

(v) Powers

Powers are continued multiplication,

$$\text{e.g. } a^3 = aaa = a \times a \times a$$
$$a^m = a \times a \times a \times a \ldots (m \text{ factors})$$
$$a^n = a \times a \times a \times a \ldots (n \text{ factors})$$

Therefore $a^m \times a^n = a^{(m+n)}$ (1)

We can write, as a convenience,

$$\frac{1}{a^n} \text{ in the form } a^{-n}$$

where the $-n$ is not a true index (or exponential) but merely a way of writing $1/a^n$.

Therefore $\frac{a^m}{a^n} = a^m \times a^{-n} = a^{m-n}$ (2)

which indicates that the $-n$ can be treated as though it were a true index.

The following can also be derived :

$$\frac{a^m}{a^m} = a^{(m-m)} = a^o \quad (3)$$

$$\text{but } \frac{a^m}{a^m} = 1$$

$$\text{Therefore } a^o = 1 \quad (4)$$

$$(a^m)^n = a^{m \times n} = a^{mn} \quad (5)$$

$$(ab)^n = a^n b^n \quad (7)$$

$$\left(\frac{b}{a}\right)^{-n} = \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \quad (8)$$

$$(-a)^n = (-1)^n \times a^n \quad (9)$$

$$= +a^n \text{ if } n \text{ is even} \quad (9a)$$

$$\text{or } = -a^n \text{ if } n \text{ is odd} \quad (9b)$$

These identities hold even when m and n are negative or fractions.

(vi) Roots

$$\sqrt{a \times a} = a \text{ or } \sqrt{a^2} = a \quad (10)$$

$$\sqrt[3]{a \times a \times a} = a \text{ or } \sqrt[3]{a^3} = a \quad (11)$$

We may adopt as a convenience the form

$$\sqrt{a} = a^{\frac{1}{2}} \quad (12)$$

$$\sqrt[3]{a} = a^{1/3} \quad (13)$$

$$\sqrt[n]{a} = a^{1/n} \quad (14)$$

This may be extended to include

$$\sqrt[n]{a^m} = (a^m)^{1/n} = a^{m/n} \quad (15)$$

so that $a^{m/n}$ is the nth root of a^m.

Note that $1/\sqrt[3]{a} = \sqrt[3]{1/a}$; $1/\sqrt[n]{a} = \sqrt[n]{1/a}$ (16)

(vii) Brackets and simple manipulations

$$a(a + b) = a \times a + a \times b = a^2 + ab \quad (17)$$

$$x(a + b - c) = xa + xb - xc \quad (18)$$

$$-x(a + b) = -xa - xb \quad (19)$$

$$-x(a - b) = -xa + xb = x(b - a) \quad (20)$$

$$-[(a - b) - (c + d)] = -(a - b) + (c + d) \quad (21)$$

$$= -a + b + c + d \quad (21a)$$

$$= (b + c + d) - a \quad (21b)$$

$$\frac{ax + bx}{cx + dx} = \frac{(a + b)x}{(c + d)x} = \frac{a + b}{c + d} \times \frac{x}{x} = \frac{a + b}{c + d} \quad (22)$$

$$(a + b)^2 = (a + b)(a + b) = a(a + b) + b(a + b) \quad (23)$$

$$= a^2 + 2\,ab + b^2 \quad (23a)$$

$$(a - b)^2 = (a - b)(a - b) = a(a - b) - b(a - b) \quad (24)$$

$$= a^2 - 2\,ab + b^2 \quad (24a)$$

$$(a + b)(a - b) = a(a - b) + b(a - b) = a^2 - b^2 \quad (25)$$

$$(a + b)(x + y) = ax + ay + bx + by \quad (26)$$

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \quad (27)$$

$$\frac{a}{-b} = -\frac{a}{b}; \quad \frac{-12ac}{4a} = -3c; \quad \frac{a^5}{a^2} = a^3 \quad (28)$$

$$+\frac{a}{b} = \frac{-a}{-b} = -\frac{-a}{b} = -\frac{a}{-b} \quad (29)$$

$$a \times 0 = 0; \quad \frac{a}{0} = \text{infinity*} = \infty \quad (30)$$

(note that it is not possible to divide by 0 in algebraic computations).

*__Infinity__ may be described as a quantity large without limit.

$$\frac{a}{c} \pm \frac{b}{d} = \frac{ad \pm bc}{cd}\,; \qquad \frac{a}{c} \pm \frac{b}{c} = \frac{a \pm b}{c}\,; \qquad \frac{a}{c} \pm \frac{a}{d} = \frac{a(d \pm c)}{cd} \tag{31}$$

The sign $\pm$ means either plus or minus. When $\pm$ and/or $\mp$ signs are used on both sides of an equation, the upper signs in both cases are to be taken in conjunction as one case, while the lower signs are to be taken as the other case.

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}\,; \qquad \frac{a}{b} = \frac{ac}{bc} \tag{32}$$

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc} \tag{33}$$

$$\frac{x}{y} - \frac{a+b}{c+d} = \frac{x(c+d)}{y(c+d)} - \frac{y}{y}\left(\frac{a+b}{c+d}\right) \tag{34a}$$

$$= \frac{x(c+d) - y(a+b)}{y(c+d)} \tag{34b}$$

(viii) Factoring—Examples

$$6ab + 3ac = 3a(2b + c) \tag{35}$$

$$x^2 - 7xy + 12y^2 = (x - 4y)(x - 3y) \tag{36}$$

$$2x^2 + 7x + 6 = (2x + 3)(x + 2) \tag{37}$$

$$x^2y - 4y^3 = y(x^2 - 4y^2) = y(x + 2y)(x - 2y) \tag{38}$$

(ix) Proportion

(1) If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a}{c} = \frac{b}{d}$ (39)

also $\frac{a}{b} - \frac{c}{d} = 0$ therefore $\frac{ad - bc}{bd} = 0$ (40)

from which $ad - bc = 0$ and thus $ad = bc$ (41)

(2) If $\frac{a}{b} = \frac{c}{d}$ and also $\frac{e}{f} = \frac{g}{h}$,

then $\frac{ae}{bf} = \frac{cg}{dh}$ (42)

(x) Variation

If $y = kx$, then $y \propto x$

i.e. y is directly proportional to x.

If $y = \frac{k}{x}$, then $y \propto \frac{1}{x}$

i.e. y is inversely proportional to x.

If $y = kxz$, then y varies jointly as x and z.

If $y = k\frac{x}{z}$, then y varies directly as x and inversely as z

(xi) Inequalities

The letter symbols below are positive and finite.

If $a > b$ then $a + c > b + c$, $\quad b < a$ (43a)

$a - c > b - c$, $c - a < c - b$ (43b)

$ac > bc$, $\quad bc < ac$ (43c)

$\frac{a}{c} > \frac{b}{c}$, $\quad \frac{c}{a} < \frac{c}{b}$ (43d)

If $a - c > b$ then $a > b + c$ (44)

If $a > b$ and $c > d$

then $a + c > b + d$, and $ac > bd$ (45)

(xii) Functions

We may describe $3x + 4$ as " a function of x " because its value depends upon the value of x. This is usually written as

$$F(x) = 3x + 4.$$

Other typical functions of x are

$$2x^2 + 3x + 5 \; ; \; x(x^2 + 3x) \; ;$$
$$\cos x \; ; \; \log x \; ; \; 1/x.$$

In such functions, x is called the " independent variable." It is usual to write $F(a)$ as meaning " $F(x)$ where $x = a$."

(xiii) Equations

An equation is a statement of conditional equality between two expressions containing one or more symbols representing unknown quantities. The process of determining values of the unknowns which will satisfy the equation is called solving the equation.

An **Identical Equation** is one which holds for all values of its letter symbols.

A **Linear Equation** is one in which, after getting rid of fractions, the independent variable only occurs in the first degree (e.g. x).

Example : $y = 5x + 3$.

A **Quadratic Equation** is one in which, after getting rid of fractions, the independent variable occurs in the second degree (e.g. x^2) but not in higher degree.

Example : $y = 4x^2 + 5x + 3$.

A quadratic equation in one unknown has two roots, although both may be complex (i.e. with an imaginary term).

A **Cubic Equation** is one in which, after getting rid of fractions, the independent variable occurs in the third degree (e.g. x^3) but not in higher degree.

Example : $y = 3x^3 + 4x^2 + 2x + 5$.

Note : x and y are usually taken as unknowns ; a, b, c and d as known constants.

Rules for solution of equations

1. The same quantity may be added to (or subtracted from) both sides.
   Example : If $x + 3 = 2$; then $x + 5 = 4$ and $x + 1 = 0$.
2. A term can be moved from one side to the other provided that its sign is changed.
   Example : If $a = b$; then $a - b = 0$.
3. All signs in the equation may be changed together.
   Example : If $x - a = y - b$; then $a - x = b - y$.
   This is equivalent to multiplication throughout by $- 1$.
4. Both sides can be multiplied (or divided) by the same quantity.
   Example : If $x + 2 = 5$; then $2x + 4 = 10$.
5. The reciprocal of one side is equal to the reciprocal of the other.
   Example : If $x = a$; then $\frac{1}{x} = \frac{1}{a}$.
6. Terms can be replaced by terms that are equal in value.
7. Both sides can be raised to the same power.
   Example : If $x = a$; then $x^2 = a^2$.
8. Both sides can be replaced by the same root of the original.
   Example : If $x = a$; then $\sqrt[3]{x} = \sqrt[3]{a}$, and

in general, $\sqrt[n]{x} = \sqrt[n]{a}$ if n is odd, but if n is even we must write

$$\sqrt[n]{x} = \pm \sqrt[n]{a}$$

which means that

$$\text{either } \sqrt[n]{x} = \sqrt[n]{a} \text{ or } \sqrt[n]{x} = -\sqrt[n]{a}.$$

In such cases two roots are obtained, and **both** should be tested in the original equation.

Warning

If both sides of an equation are squared, or if both sides are multiplied by a term containing the unknown, a new root may be introduced.

Solution of equations

(1) **Linear equations with one unknown**

Example: $ax + b = 0$.

Solution: $x = -b/a$. (46)

(2) **Linear equations with two unknowns**

Any linear relation between two variables, x and y, can be written in the general form

$$ax + by + c = 0 \tag{47}$$

or (provided that b is not zero) in the alternative form

$$y = mx + n \tag{48a}$$

This type of equation is not limited to one or two solutions, but has a corresponding value of y for every possible value of x. It is very helpful to plot the value of y against the value of x on squared paper—see Sect. 5(i). With any equation of this type, the graph will be a straight line, and it is only necessary to determine

(1) one point on the line

(2) the slope of the line at any point.

The most convenient point is usually $x = 0$, and in eqn. (48a)
this will give $y = n$
or in eqn. (47) this will give $y = -c/b$

The slope of the line is given by the difference of the y values of two points, divided by the difference of their x values.

In eqn. (47) the slope is $-a/b$
while in eqn. (48a) the slope is m.

A particular form of eqn. (47) is

$$\frac{x}{a} + \frac{y}{b} = 1 \tag{48b}$$

and in this case the line cuts the x axis at $x = a$ and cuts the y axis at $y = b$. The slope is equal to $-b/a$.

An equation of the form

$$\frac{a}{x} + \frac{b}{y} = c \tag{49}$$

may be solved by regarding $1/x$ and $1/y$ as the unknowns, then following a similar procedure as for an equation in x and y, and solving for $1/x$ and $1/y$.

(3) **Simultaneous linear equations (two unknowns)**

$$\left.\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}\right\} \qquad x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \tag{50a}$$

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} \tag{50b}$$

provided that $(a_1b_2 - a_2b_1)$ is not zero.

Alternatively the solution may be derived by determining x in terms of y from the first equation, and then substituting in the second.

Checking solutions:

After any solution has been found, particularly with more than one solution, it is highly desirable to check these in the original equation.

(4) **Three simultaneous equations (three unknowns)**

Given
$$\begin{cases} ax + by + cz + d = 0. \\ a_1x + b_1y + c_1z + d_1 = 0 \\ a_2x + b_2y + c_2z + d_2 = 0 \end{cases}$$

Then

$$\left.\begin{aligned} x &= \frac{d(b_2c_1 - b_1c_2) + d_1(bc_2 - b_2c) + d_2(b_1c - bc_1)}{a(b_1c_2 - b_2c_1) + a_1(b_2c - bc_2) + a_2(bc_1 - b_1c)} \\ y &= \frac{d(a_1c_2 - a_2c_1) + d_1(a_2c - ac_2) + d_2(ac_1 - a_1c)}{a(b_1c_2 - b_2c_1) + a_1(b_2c - bc_2) + a_2(bc_1 - b_1c)} \\ z &= \frac{d(a_2b_1 - a_1b_2) + d_1(ab_2 - a_2b) + d_2(a_1b - ab_1)}{a(b_1c_2 - b_2c_1) + a_1(b_2c - bc_2) + a_2(bc_1 - b_1c)} \end{aligned}\right\} \quad (51)$$

It will be noticed that the three denominators are identical.

(5) **Quadratic equations**

$$(x - a)(x + b) = 0\,;\; x = a \text{ or } x = -b \quad (52)$$

$$ax^2 + bx + c = 0\,;\; x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (53)$$

Note that in eqn. (53)
when $b^2 = 4ac$, the two roots are equal
when $b^2 - 4ac$ is positive, the two roots are real.
when $b^2 - 4ac$ is negative, the roots are imaginary.

(6) **Quadratic equations with two variables**

Example: $y = ax^2 + bx + c$.

This type of equation is not limited to one or two solutions, but has a corresponding value of y for every possible value of x. It is helpful to plot part of the curve on squared paper—see Sect. 5(i). The curve may cut the x axis at two points, or it may touch at one point, or it may not touch it at all. Let $y = 0$, then

$$ax^2 + bx + c = 0$$

and the points at which the curve cuts the x axis will be

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (54)$$

If $b^2 > 4ac$, the curve will cut at two points.
If $b^2 = 4ac$, the curve will touch at one point.
If $b^2 < 4ac$, the curve will not cut the x axis.

(xiv) Formulae or laws

A formula is a law, or rule, generally in connection with some scientific relationship, expressed as an equation by means of letter symbols (variables) and constants.

For example, Ohm's Law states that $E = RI$ where each of the letter symbols has a precise meaning. If we know any two of the variables, we can determine the third,

$$R = \frac{E}{I} \text{ and } I = \frac{E}{R}$$

Another example is

$$X_c = \frac{1}{2\pi fC}$$

which gives X_c for any desired values of f and C. Note that 2π is a constant.

All formulae or laws may be rearranged in accordance with the rules for the solution of equations, so as to give the value of any variable in terms of the others.

(xv) Continuity and limits

Some functions are "continuous," that is to say they are smooth and unbroken when plotted as curves. Other functions are said to be "discontinuous" if at some value of x the value of y is indeterminate or infinite, or there is a sharp angle in the plotted value of $y = F(x)$. Examples of points of discontinuity are:

(1) $y = 1/x$ is discontinuous at $x = 0$.

(2) $y = 10^{1/(x-1)}$ is discontinuous at $x = 1$.

Even when a function is discontinuous at one or more points, it may be described as continuous within certain limits.

It frequently happens that a function approaches very closely to a limiting value, although it never quite reaches it for any finite values of the independent variable. For example, the voltage gain of a resistance coupled amplifier is given by

$$A = \mu R_L/(R_L + r_p)$$

and it is required to find the limiting value of A when R_L is made very great.

The formula may be put in the form

$$A = \mu \frac{1}{1 + \frac{r_p}{R_L}} \tag{55}$$

and as R_L is made very much greater than r_p, the value of r_p/R_L becomes very small although it never actually reaches zero. We may express this in the form

$$\underset{R_L \to \infty}{Lt} \left(\frac{r_p}{R_L}\right) = 0 \tag{56}$$

which may be stated " the limit of (r_p/R_L), as R_L approaches infinity, is zero." The limiting value of A, as R_L approaches infinity, is therefore

$$\underset{R_L \to \infty}{Lt}\, A = \mu \tag{57}$$

(xvi) Progressions, sequences and series

A **Sequence** is a succession of terms so related that each may be derived from one or more of the preceding terms in accordance with some fixed law.

A **Series** is the sum of terms of a sequence.

Arithmetical Progression is a sequence, each term of which (except the first) is derived from the preceding term by the addition of a constant number.

General form : a, $(a + d)$, $(a + 2d)$, $(a + 3d)$
Example : 2, 5, 8, 11, etc.
(here $a = 2$ and $d = 3$).

The nth term $= a + (n - 1)d$ (58)

The sum of n terms is $S = \frac{1}{2}n[2a + (n - 1)d]$ (59)

When three numbers are in Arithmetical Progression, the middle number is called the " arithmetical mean." The **arithmetical mean** between a and b is $\frac{1}{2}(a + b)$.

Geometrical Progression is a sequence, each term of which (except the first) is derived from the preceding term by multiplying it by a constant ratio (r).

General form :	a,	ar,	ar^2	ar^3,	
Examples :	3,	6 ,	12 ,	24 ,	$(r = 2)$
	4,	-2,	$+1$,	$-\frac{1}{2}$	$(r = -\frac{1}{2})$

With the general form above,

$$\text{the } n\text{th term} = ar^{n-1} \tag{60}$$

and the sum of the first n terms is

$$S = a\left(\frac{r^n - 1}{r - 1}\right) \tag{61}$$

When three numbers are in Geometrical Progression, the middle number is called the Geometrical Mean. The **Geometrical Mean** of two numbers a and b is $\sqrt{ab}$.

If the ratio r is less than unity, the terms become progressively smaller, and the sum of a very large number of terms approaches

$$\underset{n \to \infty}{S} = \frac{a}{1 - r} \tag{62}$$

Harmonic Progression : The terms a, b, c, etc. form a harmonic sequence if their reciprocals

$$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \text{ etc.}$$

form an arithmetical sequence.

Example : 1, $\frac{1}{2}$, 1/3, $\frac{1}{4}$, 1/5 is a Harmonic Progression
because 1, 2, 3 , 4, 5 is an Arithmetical Progression

The Harmonic Mean between a and b is $\frac{2ab}{a + b}$.

Note that the Arithmetical Mean between two numbers is greater than the Geometrical Mean, which in turn is greater than the Harmonic Mean.

With any form of progression or sequence, we are often concerned with the sum of a number of terms of which the general term is given. It is possible to write this sum in a shortened form, for example

$$1 + 2 + 3 + 4 + \ldots + k + \ldots + n = \sum_{k=1}^{k=n} k$$

where the Greek letter capital sigma is used to indicate the sum of a number of terms ; k is merely the general term ; and the values of k are to be taken from $k = 1$ (beneath sigma) to $k = n$ (written above sigma).

(xvii) Logarithmic and exponential functions

If $a^x = N$

then x is the logarithm to the base a of the number N. This may be put in the form

$$x = \log_a N$$

where the base a may be any positive number except 1 or 0.

The two principal systems of logarithms are

(1) The Naperian (or natural) system, using the base $e = 2.718\,28\ldots$ (preferably written with the Greek ϵ), and

(2) The Briggs (or common) system, using the base 10.

Only one set of tables is required, for it is possible to convert a logarithm to one base (b) into a logarithm to any other base (a) :

$$\log_a y = \log_b y \times \log_a b$$

If it is required to find the logarithm to the base ϵ, given the logarithm to the base 10,

$$\log_\epsilon y = \log_{10} y \times \log_\epsilon 10 \quad (63a)$$

$$= \log_{10} y \times 2.3026 \quad (63b)$$

$$\log_{10} y = \log_\epsilon y \times 0.4343 \quad (63c)$$

Some properties of ϵ

The value of ϵ is given by the right hand side of eqn. (86).

Values shown below in brackets are to four decimal places.

$\epsilon = 2.71828\ (= 2.7183)$

$1/\epsilon = 0.367879\ (= 0.3679)$

$\log_{10} \epsilon = 0.43429\ (= 0.4343)$

$\log_\epsilon 10 = 2.30259\ (= 2.3026)$

$\log_\epsilon 10 = 1/\log_{10} \epsilon$

$\log_{10} \epsilon^n = n \times 0.43429\ (= n \times 0.4343)$

Some manipulations with logarithmic functions

$$\log a/b = \log a - \log b \quad (64)$$

$$\log 1/a = -\log a \quad (65)$$

$$\log y^n = n \times \log y \quad (66)$$

$$\log y^{-1} = -\log y \quad (67)$$

$$\log y^{m/n} = (m/n) \times \log y \quad (68)$$

$$\log \sqrt{y} = \log y^{1/2} = (1/2) \log y \quad (69)$$

$$\log \sqrt[3]{y} = \log y^{1/3} = (1/3) \log y \quad 70)$$

To find the cube root of 125—

$\sqrt[3]{125} = (125)^{1/3}$

Therefore $\log (125)^{1/3} = (1/3) \log 125 = (1/3)(2.0969) = 0.699$.

Then antilog $0.699 = 5.00$ (from tables).

$$\log \sqrt[4]{y^3} = \tfrac{3}{4} \log y \quad (71)$$

$$\log abc = \log a + \log b + \log c \quad (72)$$

$$\log (ab/cd) = \log a + \log b - \log c - \log d \quad (73)$$

$$\log (a^m b^n c^p) = m. \log a + n. \log b + p. \log c \quad (74)$$

$$\log (ab^m/c^n) = \log a + m. \log b - n. \log c \quad (75)$$

$$\log (a^2 - b^2) = \log [(a + b)(a - b)] = \log (a + b) + \log (a - b) \quad (76)$$

$\log \sqrt{a^2 - b^2} = \frac{1}{2} \log (a + b) + \frac{1}{2} \log (a - b)$ (77)

Logarithmic Functions are closely related to **Exponential Functions,** and any equation in one form may be put into the other form. If the curves are plotted, the two will be the same.

Example: Exponential form $y = r^x$
Logarithmic form $x = \log_r y$

Numerical example:
If $r = 10$ and $x = 3$
Then $y = 10^3 = 1000$.
This may be handled by the logarithmic form of the equation,

$$x = \log_r y = \log_{10} 1000 = 3 \text{ as before.}$$

Logarithmic decrement: If the equation is of the form

$$y = a\epsilon^{-bx}$$

where $(-b)$ is negative, the value of y decreases as x is increased, and $(-b)$ is called the Logarithmic Decrement.

(xviii) Infinite series

It was noted, when dealing with Geometrical Progression, that it is possible to take the limit of the sum of a very large number of terms, as the number approaches infinity, provided that the terms become progressively smaller by a constant ratio. Such a series is called " convergent " and is defined as having a finite limit to the sum to infinity. Infinite series which do not comply with this definition may be " divergent " (these are not considered any further) or else they may be " oscillating."

(a) Binomial series:

$$1 + mx + \frac{m(m-1)}{1.2}x^2 + \frac{m(m-1)(m-2)}{1.2.3}x^3 + \text{etc.} \quad (78)$$

The nth term is

$$a_n = \frac{m(m-1)(m-2)\ldots(m-n+2)}{1.2.3\ldots(n-1)}x^{n-1} \quad (79)$$

The denominator is usually written in the form

$$(n-1)\,!$$

which is called " factorial $(n-1)$."

This Binomial Series is convergent, provided that x is numerically less than 1.

(b) Binomial theorem:

Case 1:

$$(1 \pm x)^m = 1 \pm mx + \frac{m(m-1)\,x^2}{2\,!} \pm \frac{m(m-1)(m-2)x^3}{3\,!} + \ldots$$
$$\pm \frac{m(m-1)\ldots(m-n+2)x^{n-1}}{(n-1)\,!} + \ldots \quad (80)$$

which holds for all values of x if m is a positive integer, and for all values of m provided that x is numerically less than 1.

The Binomial Theorem is useful in certain approximate calculations. If x is small compared with 1, and m is reasonably small,

$$\left.\begin{aligned} (1+x)^m &\approx 1 + mx \\ (1-x)^m &\approx 1 - mx \\ (1+x)^{-m} &\approx 1 - mx \\ (1-x)^{-n} &\approx 1 + mx. \end{aligned}\right\} \quad (81)$$

To a closer approximation (taking three terms),

$$\left.\begin{aligned} (1+x)^m &\approx 1 + mx + \tfrac{1}{2}\,m(m-1)\,x^2 \\ (1-x)^m &\approx 1 - mx + \tfrac{1}{2}\,m(m-1)\,x^2 \\ (1+x)^{-m} &\approx 1 - mx + \tfrac{1}{2}\,m(m+1)\,x^2 \\ (1-x)^{-m} &\approx 1 + mx + \tfrac{1}{2}\,m(m+1)\,x^2 \end{aligned}\right\} \quad (82)$$

Numerical example: To find the cube root of 220.

$$\sqrt[3]{220} = (216 + 4)^{1/3} = \left\{216\left(1 + \frac{4}{216}\right)\right\}^{1/3} = 6\left(1 + \frac{1}{54}\right)^{1/3}.$$

Applying the approximation from the Binomial Theorem,

$$(1 + x)^m \approx 1 + mx$$

to the evaluation of the quantity above, we may make $x = 1/54$ and $m = 1/3$, from which

$$\left(1 + \frac{1}{54}\right)^{1/3} \approx 1 + \frac{1}{3}\cdot\frac{1}{54} \approx 1 + \frac{1}{162} \approx 1.006\,17$$

Therefore $\sqrt[3]{220} \approx 6 \times 1.006\,17 \approx 6.037$.

Case 2 :

We can also expand $(a + x)^m$, which is convergent when x is numerically less than a:

$$(a \pm x)^m = a^m \pm ma^{m-1}x + \frac{m(m-1)}{2\,!}a^{m-2}x^2 \pm \frac{m(m-1)(m-2)}{3\,!}a^{m-3}x^3 + \ldots \quad (83)$$

Approximation :

$$\frac{1}{a+1} = (a+1)^{-1} \approx \frac{1}{a}\left(1 - \frac{1}{a}\right) \text{ when } a > 1 \quad (84)$$

(c) Exponential series

From the Binomial Theorem, putting $x = 1/n$ and $m = nx$, we may derive

$$\left(1 + \frac{1}{n}\right)^{nx} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^{n-1}}{(n-1)!} + \ldots \quad (85)$$

When $x = 1$ this becomes

$$\left(1 + \frac{1}{n}\right)^{n} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots + \frac{1}{(n-1)!} + \ldots \quad (86)$$

The right hand side of this equation is the value ϵ, which is equal to 2.71828 (to five places of decimals). Taking the xth power of each side of this equation,

$$\left(1 + \frac{1}{n}\right)^{nx} = \epsilon^x$$

Therefore $\epsilon^x = \left(1 + \frac{1}{n}\right)^{nx}$

and
$$\epsilon^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^{n-1}}{(n-1)!} + \ldots \quad (87)$$

which is called the Exponential Series.

(d) Logarithmic series

The logarithmic series is the expansion of $\log_\epsilon (1 + x)$ in ascending powers of x :

$$\log_\epsilon (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \quad (88)$$

which is convergent if x is numerically less than 1.

(e) Trigonometrical series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \quad (89)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \quad (90)$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \ldots (|x| < \pi/2) \quad (91)$$

For derivation of eqns. (89) and (90) see Sect. 6, eqns. (17) and (18).

(xix) Hyperbolic functions

These are combinations of the sum and difference of two exponential functions.

$\dfrac{\epsilon^x - \epsilon^{-x}}{2}$ is called the hyperbolic sine of x, designated by $\sinh x$

$\dfrac{\epsilon^x + \epsilon^{-x}}{2}$ is called the hyperbolic cosine of x, designated by $\cosh x$,

$\frac{\epsilon^x - \epsilon^{-x}}{\epsilon^x + \epsilon^{-x}}$ is called the hyperbolic tangent of x, designated by $\tanh x$

and similarly with the inverses

$\operatorname{cosech} x = 1/\sinh x$
$\operatorname{sech} x = 1/\cosh x$
$\coth x = 1/\tanh x$

Note $\epsilon = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots \approx 2.718\,28$ (92)

The following may be derived :

$\cosh^2 x - \sinh^2 x = 1$ (93)

$\operatorname{sech}^2 x + \tanh^2 x = 1$ (94)

$\coth^2 x - \operatorname{cosech}^2 x = 1$ (95)

$\sinh(-x) = -\sinh x$; $\cosh(-x) = \cosh x$; $\tanh(-x) = -\tanh x$ (96)

$$\sinh x = \frac{\tanh x}{\sqrt{1 - \tanh^2 x}} = \sqrt{\cosh^2 x - 1} \quad (97)$$

$$\cosh x = \frac{1}{\sqrt{1 - \tanh^2 x}} = \sqrt{\sinh^2 x + 1} \quad (98)$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \sqrt{1 - \operatorname{sech}^2 x} \quad (99)$$

$$\sinh x + \sinh y = 2 \sinh \frac{x+y}{2} \cosh \frac{x-y}{2} \quad (100)$$

$$\sinh x - \sinh y = 2 \sinh \frac{x-y}{2} \cosh \frac{x+y}{2} \quad (101)$$

$$\cosh x + \cosh y = 2 \cosh \frac{x+y}{2} \cosh \frac{x-y}{2} \quad (102)$$

$$\cosh x - \cosh y = 2 \sinh \frac{x+y}{2} \sinh \frac{x-y}{2} \quad (103)$$

$$\tanh x \pm \tanh y = \frac{\sinh(x \pm y)}{\cosh x \cosh y} \quad (104)$$

$$(\sinh x + \cosh x)^n = \cosh nx + \sinh nx \quad (105)$$

$$\sinh^{-1} x = \log_\epsilon (x + \sqrt{x^2 + 1}) = \cosh^{-1}\sqrt{x^2 + 1} \quad (106)$$

$$\cosh^{-1} x = \log_\epsilon (x + \sqrt{x^2 - 1}) = \sinh^{-1}\sqrt{x^2 - 1} \quad (107)$$

$$\tanh^{-1} x = \tfrac{1}{2} \log_\epsilon \frac{1+x}{1-x} \quad (108)$$

$$= \cosh^{-1} \frac{1}{\sqrt{1 - x^2}} = \sinh^{-1} \frac{x}{\sqrt{1 - x^2}} \quad (109)$$

$$\sinh(x \pm y) = \frac{\epsilon^{x \pm y} - \epsilon^{-(x \pm y)}}{2} = \sinh x \cosh y \pm \cosh x \sinh y \quad (110a)$$

$$\cosh(x \pm y) = \frac{\epsilon^{x \pm y} + \epsilon^{-(x \pm y)}}{2} = \cosh x \cosh y \pm \sinh x \sinh y \quad (110b)$$

$$\tanh(x \pm y) = \frac{\epsilon^{x \pm y} - \epsilon^{-(x \pm y)}}{\epsilon^{x \pm y} + \epsilon^{-(x \pm y)}} = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y} \quad (111)$$

$$\sinh 2x = 2 \sinh x \cosh x = \frac{2 \tanh x}{1 - \tanh^2 x} \quad (112)$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x = \frac{1 + \tanh^2 x}{1 - \tanh^2 x} \quad (113)$$

$$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x} \quad (114)$$

$\cosh x + \sinh x = \epsilon^x$ (115a)

$\cosh x - \sinh x = \epsilon^{-x}$ (115b)

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \quad (115c)$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots \qquad (115d)$$

(these are convergent for all real values of x).

Also in complex form (see Section 6)

$$\sinh jx = j \sin x\,;\ \cosh jx = \cos x\,;\ \tanh jx = j \tan x \qquad (116)$$

$$\sin jx = j \sinh x\,;\ \cos jx = \cosh x \qquad (117)$$

$$\sinh (x \pm jy) = \sinh x \cos y \pm j \cosh x \sin y \qquad (118)$$

Note that x and y in sin x, cos x, tan x etc. in eqns. 116 to 119 must be expressed in radians.

$$\cosh (x \pm jy) = \cosh x \cos y \pm j \sinh x \sin y \qquad (119)$$

$$\sinh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh x - 1}{2}}\,; \qquad \cosh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh x + 1}{2}} \qquad (120)$$

$$\tanh\left(\frac{x}{2}\right) = \frac{\cosh x - 1}{\sinh x} = \frac{\sinh x}{\cosh x + 1} \qquad (121)$$

(xx) General approximations

Let δ be an extremely small quantity and x be a quantity very large compared with δ, then

$$\frac{1}{1 - \delta} \approx 1 + \delta \qquad \frac{1}{1 + \delta} \approx 1 - \delta \qquad (122)$$

$$\frac{1 + \delta_1}{1 + \delta_2} \approx 1 + \delta_1 - \delta_2 \qquad (123)$$

$$(1 \pm \delta)^n \approx 1 \pm n\delta \qquad (124)$$

$$\frac{1}{(1 \pm \delta)^n} \approx 1 \mp n\delta \qquad (125)$$

where n may be integral, fractional or negative (124), (125)

$$\sqrt{1 + \delta} \approx 1 + \tfrac{1}{2}\delta \qquad \sqrt{1 - \delta} \approx 1 - \tfrac{1}{2}\delta \qquad (126)$$

$$\frac{1}{\sqrt{1 + \delta}} \approx 1 - \tfrac{1}{2}\delta \qquad \frac{1}{\sqrt{1 - \delta}} \approx 1 + \tfrac{1}{2}\delta \qquad (127)$$

$$(1 + \delta)^2 \approx 1 + 2\delta \qquad (1 - \delta)^2 \approx 1 - 2\delta \qquad (128)$$

$$(x \pm \delta)^n \approx x^n \pm nx^{n-1}\delta \qquad (129)$$

Eqn. (129) is used in Sect. 1(v)b for approximations with powers and roots.

$$\sqrt{x(x + \delta)} \approx x + \tfrac{1}{2}\delta \qquad \sqrt{x(x - \delta)} \approx x - \tfrac{1}{2}\delta \qquad (130)$$

$$(1 + \delta_1)(1 \pm \delta_2) \approx 1 + \delta_1 \pm \delta_2 \qquad (131)$$

$$(1 + \delta_1)(1 + \delta_2)(1 + \delta_3) \approx 1 + \delta_1 + \delta_2 + \delta_3 \qquad (132)$$

where δ_1, δ_2 and δ_3 are all extremely small quantities.

$$\epsilon^{\delta} \approx 1 + \delta \qquad \epsilon^{-\delta} \approx 1 - \delta \qquad (133)$$

$$\epsilon^{t/RC} \approx 1 + t/RC \qquad \epsilon^{-t/RC} \approx 1 - t/RC \qquad (134)$$

Eqn. (134) has an error less than 0.6% when t/RC does not exceed 0.1.

$$\log_\epsilon (x \pm \delta) \approx \log_\epsilon x \pm \frac{\delta}{x} - \tfrac{1}{2}\left(\frac{\delta}{x}\right)^2 \qquad (135)$$

$$\log_\epsilon (1 \pm \delta) \approx \delta - \tfrac{1}{2}\delta^2 \qquad (136)$$

$$\sinh \delta \approx \delta \qquad \cosh \delta \approx 1 \qquad \tanh \delta \approx \delta \qquad (137)$$

$$\sinh^{-1}\delta \approx \delta \qquad \tanh^{-1}\delta \approx \delta \qquad (138)$$

$$\sinh (x + \delta) \approx \sinh x + \delta \cosh x\,;\ \sinh (x - \delta) \approx \sinh x - \delta \cosh x \qquad (139)$$

$$\cosh (x + \delta) \approx \cosh x + \delta \sinh x\,;\ \cosh (x - \delta) \approx \cosh x - \delta \sinh x \qquad (140)$$

$$\tanh (x + \delta) \approx \tanh x + \delta \operatorname{sech}^2 x\,;\ \tanh (x - \delta) \approx \tanh x - \delta \operatorname{sech}^2 x \qquad (141)$$

When L is a large quantity

$$\sinh L \approx \tfrac{1}{2}\epsilon^L \qquad \cosh L \approx \tfrac{1}{2}\epsilon^L \qquad \tanh L \approx 1 \qquad (142)$$

Trigonometrical relationships

When δ is an extremely small quantity, so that an angle of δ radians is a very small angle, and x is an angle very large compared with δ,

$\sin \delta \approx \delta \qquad \cos \delta \approx 1 \qquad \tan \delta \approx \delta$ (143)

$\sin^{-1}\delta \approx \delta \qquad \cos^{-1}\delta \approx \tfrac{1}{2}\pi(4K - 1) + \delta \qquad \tan^{-1}\delta \approx \delta$ (144)

where K is any integer. See Sect. 3(iii) for inverse functions.

$\sin(x + \delta) \approx \sin x + \delta \cos x \qquad \sin(x - \delta) \approx \sin x - \delta \cos x$ (145)

$\cos(x + \delta) \approx \cos x - \delta \sin x \qquad \cos(x - \delta) \approx \cos x + \delta \sin x$ (146)

$\tan(x + \delta) \approx \tan x + \delta/\cos^2 x \qquad \tan(x - \delta) \approx \tan x - \delta/\cos^2 x$ (147)

SECTION 3 : GEOMETRY AND TRIGONOMETRY

(i) Plane figures (ii) Surfaces and volumes of solids (iii) Trigonometrical relationships.

(i) Plane figures

Angles

Two angles are complementary when their sum is equal to a right angle (90°).
Two angles are supplementary when their sum is equal to two right angles (180°).
The three angles of a triangle are together equal to two right angles (180°)

2π radians $= 360°$
π radians $= 180°$
1 radian $\approx 57.29578°$
$1° \approx 0.0174533$ radian.

When an angle is measured in radians, and incorporates the sign π, it is usual to omit the word " radians " as being understood—e.g. π, 2π.

Right Angle Triangles (Fig. 6.1)

Sine :* $\frac{a}{c} = \sin A \qquad a = c \sin A$ (1)

Tangent : $\frac{a}{b} = \tan A \qquad a = b \tan A$ (2)

Cosine : $\frac{b}{c} = \cos A \qquad b = c \cos A$ (3)

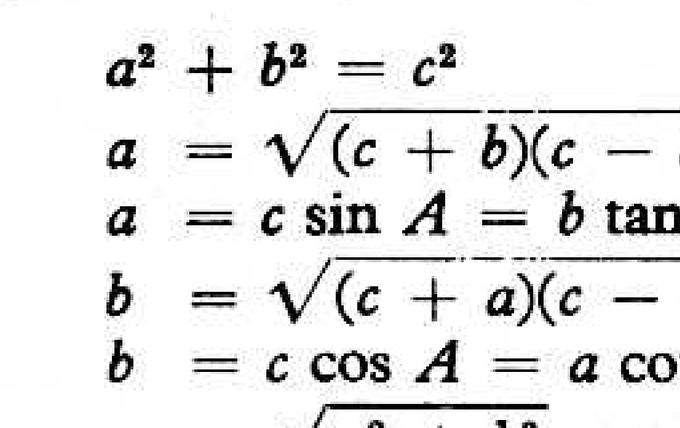


FIG. 6.1

Cosecant : $\frac{c}{a} = \operatorname{cosec} A = 1/\sin A$ (4)

Secant : $\frac{c}{b} = \sec A = 1/\cos A$ (5)

Cotangent : $\frac{b}{a} = \cot A = 1/\tan A$ (6)

c is called the hypotenuse.

$a^2 + b^2 = c^2$ (7)

$a = \sqrt{(c + b)(c - b)} = \sqrt{mc}$ (8)

$a = c \sin A = b \tan A$ (9)

$b = \sqrt{(c + a)(c - a)} = \sqrt{nc}$ (10)

$b = c \cos A = a \cot A = a/\tan A$ (11)

$c = \sqrt{a^2 + b^2} = m + n$ (12)

$c = a \operatorname{cosec} A = a/\sin A = b \sec A = b/\cos A$ (13)

Area $= \tfrac{1}{2} ab = \tfrac{1}{2} a^2 \cot A = \tfrac{1}{2} b^2 \tan A$ (14)

$= \tfrac{1}{4} c^2 \sin 2A = \tfrac{1}{2} bc \sin A = \tfrac{1}{2} ac \sin B$ (15)

The perpendicular (p) to the hypotenuse is the mean proportional (or mean geometrical progression) between the segments of the hypotenuse,

*See Sect. 3(iii) for trigonometrical relationships.

$$\frac{m}{p} = \frac{p}{n}, \quad p = \sqrt{mn} \tag{16}$$

also
$$\frac{m}{a} = \frac{a}{c}, \quad \frac{n}{b} = \frac{b}{c}. \tag{17}$$

Any triangle inscribed in a semicircle, with the diameter forming one side, is a right angle triangle.

Equilateral Triangle (Fig. 6.2a)

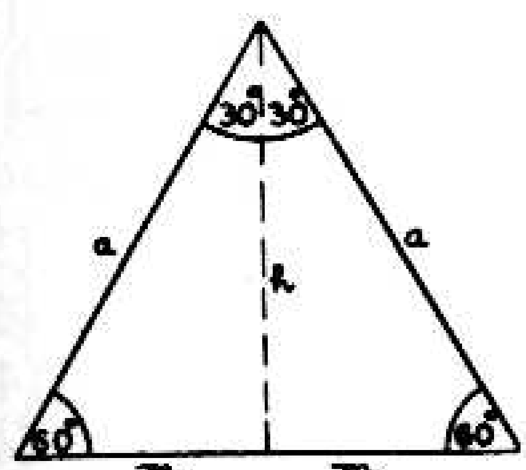


FIG. 6·2A

Each side $= a$

Each angle $= 60°$

$m = \frac{1}{2} a$

$$h = \sqrt{3}\, m = (\sqrt{3}/2)a \approx 0.866a \tag{18}$$

$$\text{area} = \tfrac{1}{2}\, ah = (\sqrt{3}/4)a^2 \approx 0.433a^2 \tag{19}$$

Any triangle (Fig. 6.2b)

$$\text{Area} = \tfrac{1}{2}\, bh = \sqrt{s(s-a)(s-b)(s-c)} \tag{20}$$

where b = base, h = height, $s = \frac{1}{2}(a + b + c)$.

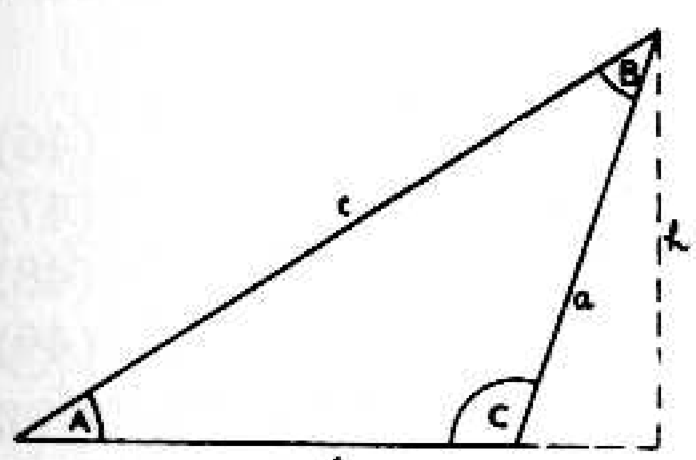


FIG. 6.2B

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \tag{21}$$

$$a^2 = b^2 + c^2 - 2bc \cos A \tag{22}$$
$$b^2 = c^2 + a^2 - 2ca \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C \tag{23}$$
$$a = b \cos C + c \cos B \tag{24}$$
$$b = c \cos A + a \cos C$$
$$c = a \cos B + b \cos A. \tag{25}$$

Rectangle (Fig. 6.3)

$$\text{Area} = ab \tag{26}$$

$$d = \text{diagonal} = \sqrt{a^2 + b^2} \tag{27}$$

Parallelogram (Fig. 6.4)

$$\text{Area} = bh = ab \sin C\,; \quad a = c\,; \quad b = d. \tag{28}$$

$$\text{Angle } A = \text{angle } C\,; \quad \text{angle } B = \text{angle } D. \tag{29}$$

Trapezoid (Fig. 6.5).

Side d is parallel to side b.

$$\text{Area} = \tfrac{1}{2}(b + d)h. \tag{30}$$

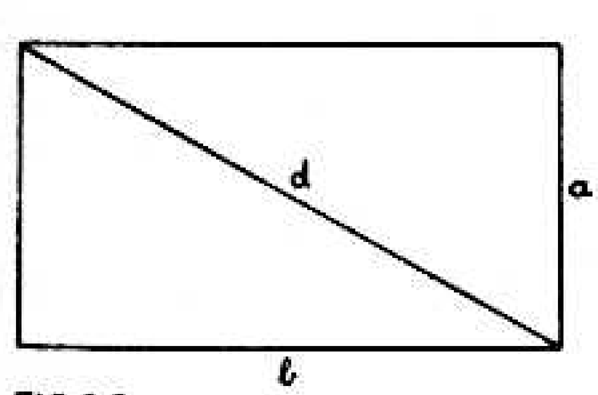


FIG. 6·3

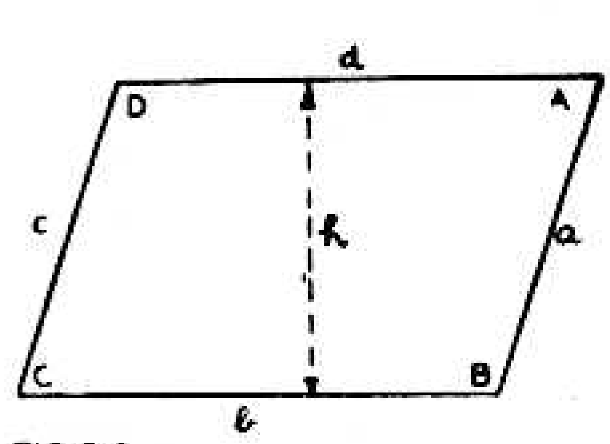


FIG. 6·4

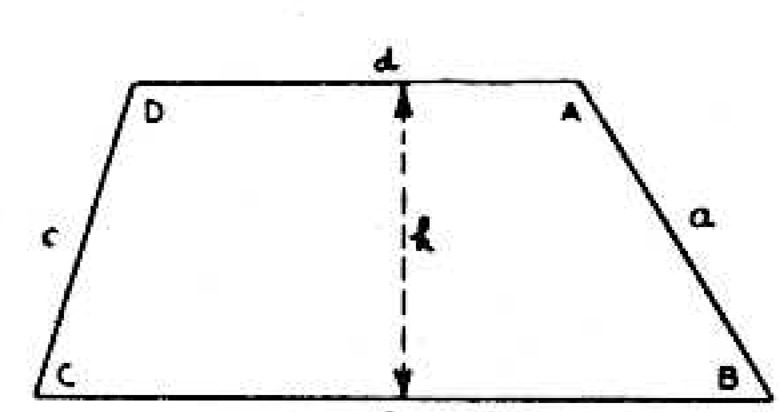


FIG. 6·5

Polygons and Quadrilaterals

To find area, divide into triangles, calculate the area of each, and add.

Circle

$$\text{Circumference} = \pi \times \text{diameter} \approx 3.1416 \times \text{diameter} \tag{31}$$
$$= 2\pi \times \text{radius} \approx 6.2832 \times \text{radius} \tag{32}$$

$$\text{Area} = \pi \times (\text{radius})^2 = \tfrac{1}{2}(\text{circumference} \times \text{radius}) \tag{33}$$
$$= (\pi/4) \times (\text{diameter})^2 \approx 0.7854\, (\text{diameter})^2 \tag{34}$$

Sector (Fig. 6.6)

$$A = \text{angle subtended at centre} = s/r \text{ radians} \tag{35}$$

$$c = 2\sqrt{r^2 - a^2} = 2\, r \sin (A/2) = 2a \tan (A/2) = 2\sqrt{2hr - h^2} \tag{36}$$

$$a = \tfrac{1}{2}\sqrt{4r^2 - c^2} = \tfrac{1}{2}\sqrt{d^2 - c^2} = r \cos (A/2) = \tfrac{1}{2}\, c \cot (A/2) \tag{37}$$

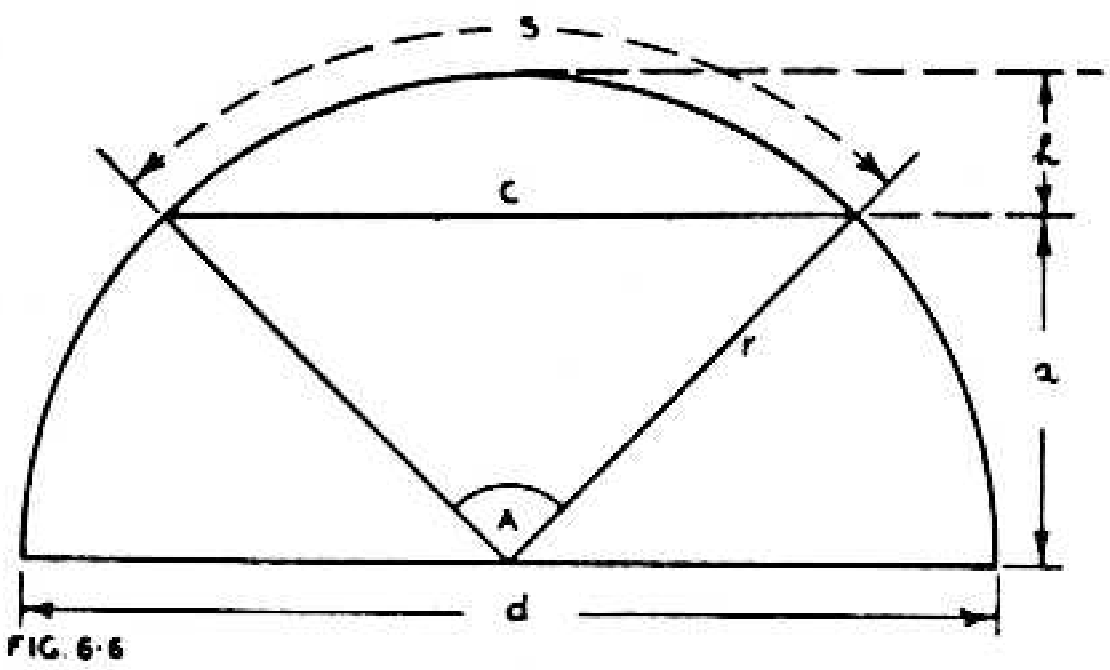


FIG. 6·6

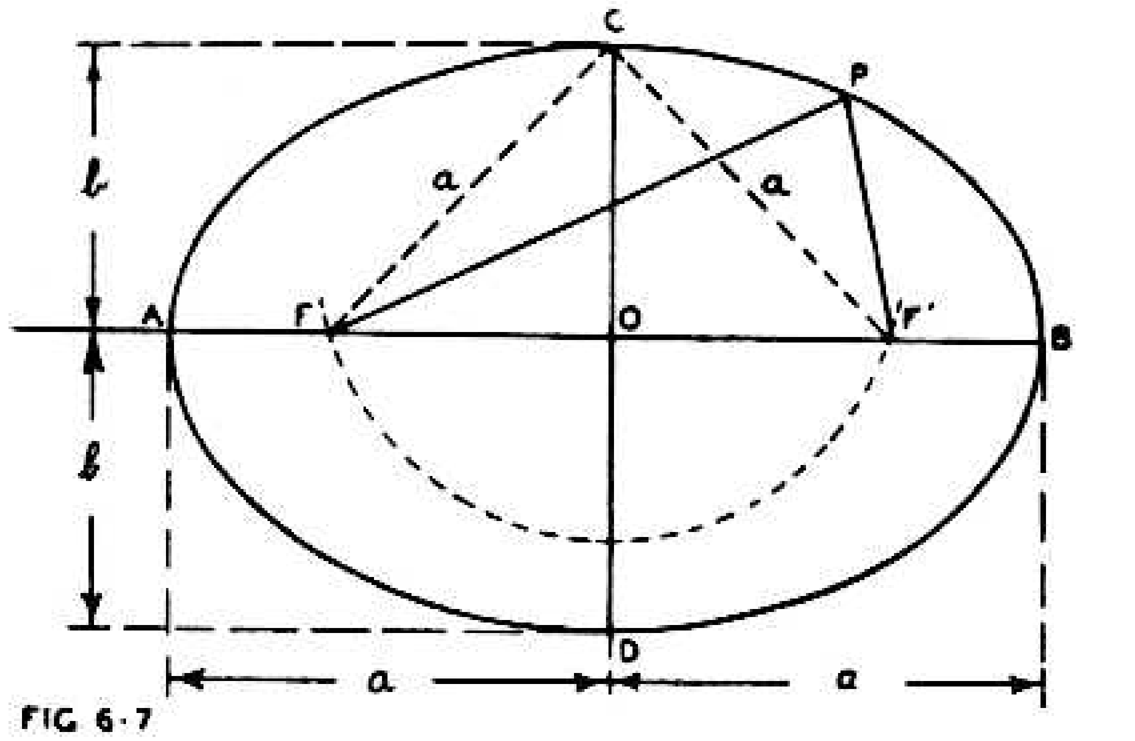


FIG. 6·7

s = length of arc $= rA$ (A in radians) (38)

$= \pi rA/180$ (A in degrees) (39)

$h = r - a = r(1 - \cos A/2)$ (40)

Area of sector $= \frac{1}{2} rs = \frac{1}{2} r^2 A$ (A in radians) (41)

$= \pi r^2 A/360$ (A in degrees) (42)

Area of segment (bounded by chord c, curve s)

$= \frac{1}{2} r^2 (A - \sin A)$ (A in radians) (43)

$= \frac{1}{2} r (s - r \sin s/r)$ (s/r in radians) (44)

$= \frac{1}{2} [r(s - c) + ch]$ for segments less than half a circle. (45)

Ellipse (Fig. 6.7)

The ellipse has two foci, F and F′, and for any point P on the perimeter,

FP + PF′ is constant = FB + BF′ = FA + AF′ (46)

Major axis $= AB = 2a$; minor axis $= CD = 2b$ (47)

Area of ellipse $= \pi ab \approx 0.7854$ major axis × minor axis (48)

Perimeter $\approx a(4 + 1.1m + 1.2m^2)$, where $m = b/a$ (49)

An ellipse may be drawn by putting a pin into the paper at each focus (F, F′), tying the ends of a short length of cotton thread around the pins leaving a slack portion in the middle, and running a pencil point around in the loop of the thread.

To find the foci, draw an arc with centre at C and radius a to intersect the X axis (AB) at F and F′.

Parabola (Fig. 6.8)

The parabola has a focus (F) and a directrix (MN) and for any point P on the parabola,

FP = PM where PM is the perpendicular to the directrix (50)

Area of segment cut off by chord PP′ = (2/3) ch. (51)

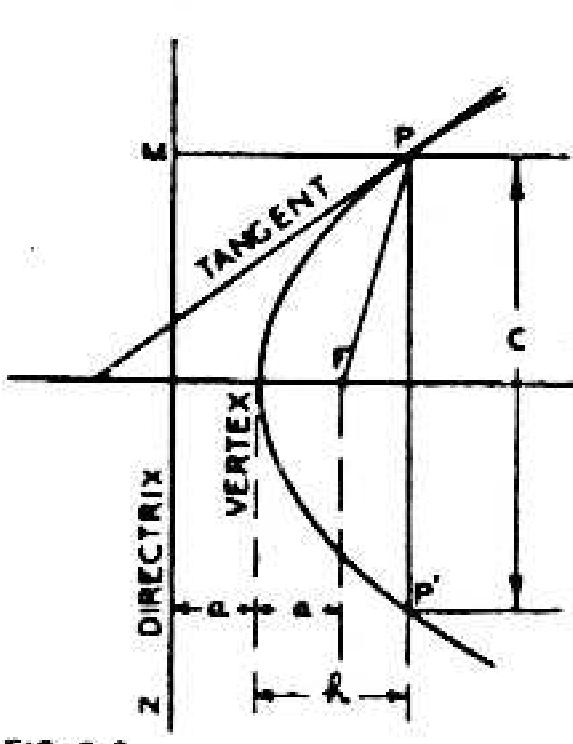


FIG. 6·8

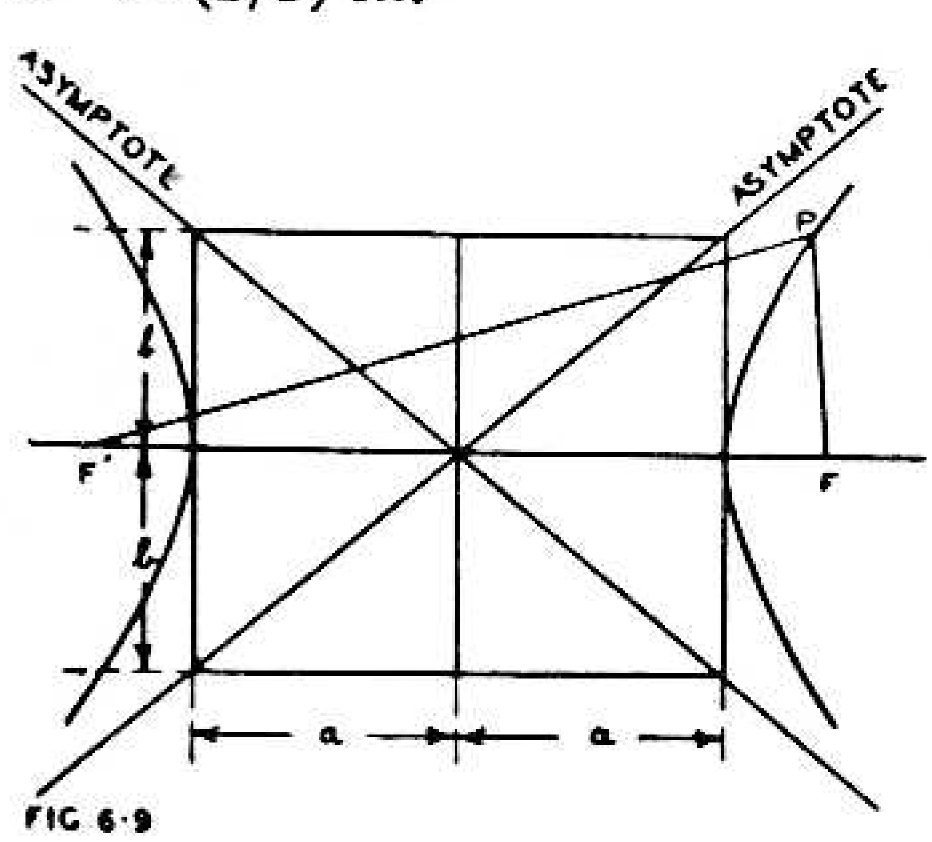


FIG. 6·9

Hyperbola (Fig. 6.9)

The hyperbola has two foci (F, F′) and two asymptotes, and for any point P on either curve,

$$F'P - FP = \text{constant} \quad (52)$$

General rules for areas

Areas bounded by straight sides may be calculated by dividing the area into triangles, calculating the area of each and adding.

Areas bounded by irregular curves may be divided into parallel strips and the area calculated by one of the following approximations :

Trapezoid rule

Area $= d(\frac{1}{2}y_1 + y_2 + y_3 + \ldots\ldots\ldots\ldots + y_{n-1} + \frac{1}{2}y_n)$ (53)

where d = width of each strip

and $y_1, y_2, y_3 \ldots . y_n$ are measured lengths of each of the equidistant parallel chords. Note that the first (y_1) and the last (y_n) do not cut the area, and may be zero if the surface is sharply curved.

Simpson's Rule

Area $= \frac{d}{3}(y_1 + 4y_2 + 2y_3 + 4y_4 + 2y_5 \ldots + 2y_{n-2} + 4y_{n-1} + y_n)$ (54)

where n must be odd

d = width of each strip

and $y_1 \ldots y_n$ are measured lengths of equidistant parallel chords.

(ii) Surfaces and volumes of solids

Cube (length of side $= a$)

Volume $= a^3$ (55)

Surface area $= 6a^2$ (56)

Length of diagonal $= a\sqrt{3}$ (57)

Rectangular prism

(length $= l$, breadth $= b$, height $= h$)

Volume $= lbh$ (58)

Surface area $= 2(lb + lh + bh)$ (59)

Diagonal $= \sqrt{b^2 + l^2 + h^2}$ (60)

Cylinder, solid right circular

(length l, radius r)

Volume $= \pi r^2 l \approx 0.7854\ d^2 l$ (61)

Area of curved portion $= 2\pi\, rl = \pi\, dl$ (62)

Area of each end $= \pi\, r^2$ (63)

Total surface area $= 2\pi\, r\, (l + r)$ (64)

Hollow cylinder, right circular

(length l, outer radius R, inside radius r)

Volume $= \pi\, l\, (R^2 - r^2)$ (65)

Any pyramid or cone

Volume = 1/3 (area of base × distance from vertex to plane of base) (66)

Sphere

Volume $= \frac{4}{3}\pi\, r^3 = \pi\, d^3/6 \approx 4.1888\ r^3 \approx 0.5236\ d^3$ (67)

Surface area $= 4\pi\, r^2 = \pi\, d^2$ (68)

(iii) Trigonometrical relationships

We have already introduced the sine, cosine and tangent of an angle, and their inverses, under the subject Angles (Fig. 6.1). The following table may readily be derived with the assistance of Fig. 6.10 :

Angle	Sine	Cosine	Tangent
0	0	1	0
30°	1/2	$\sqrt{3}/2$	$1/\sqrt{3}$
45°	$1/\sqrt{2}$	$1/\sqrt{2}$	1
60°	$\sqrt{3}/2$	1/2	$\sqrt{3}$
90°	1	0	∞ *

*Approaches infinity as angle approaches 90°.

The values for any angle between 0° and 90° may be found from Table 72, Trigonometrical Relationships, in Chapter 38 Sect. 21, or from any book of Mathematical Tables.

Angles of any magnitude

If the line OX (Fig. 6.11) revolves about O to a new position OP, the amount of rotation is the angle XOP between its original position OX and its new position OP. Such a counter-clockwise rotation is called positive, while the opposite direction of rotation is called negative.

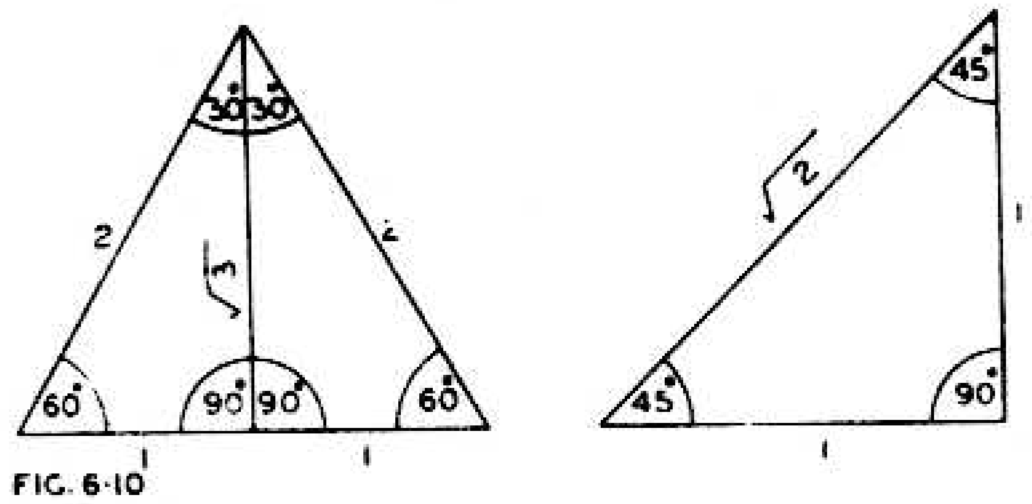


FIG. 6·10

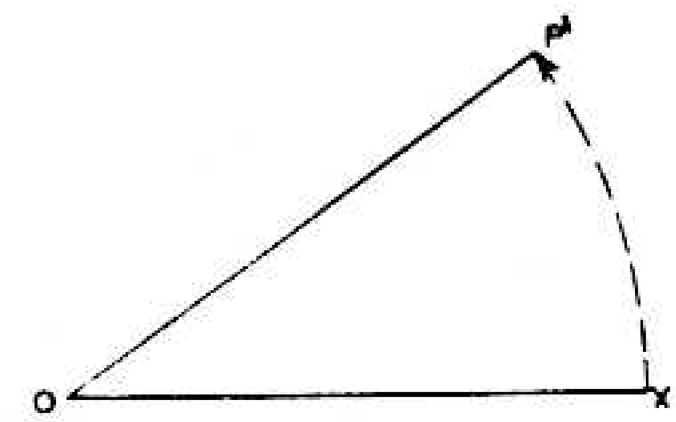


FIG. 6·11

Examples of angles in all four sectors are shown in Fig. 6.12. It will be seen that, for any angle A, the position of OP is the same for a movement of angle A in a positive direction, or for a negative movement of $(360° - A)$; for example,

$$+ 330° = -(360° - 330°) = -30°$$

In the case of angles greater than 360° we are generally only concerned with the final position of OP, so that for these cases we may subtract 360°, or any multiple of 360°, from the angle so as to give a value less than 360°. For example $390° = 360° + 30°$; $800° = 720° + 80°$; $1125° = 1080° + 45°$.

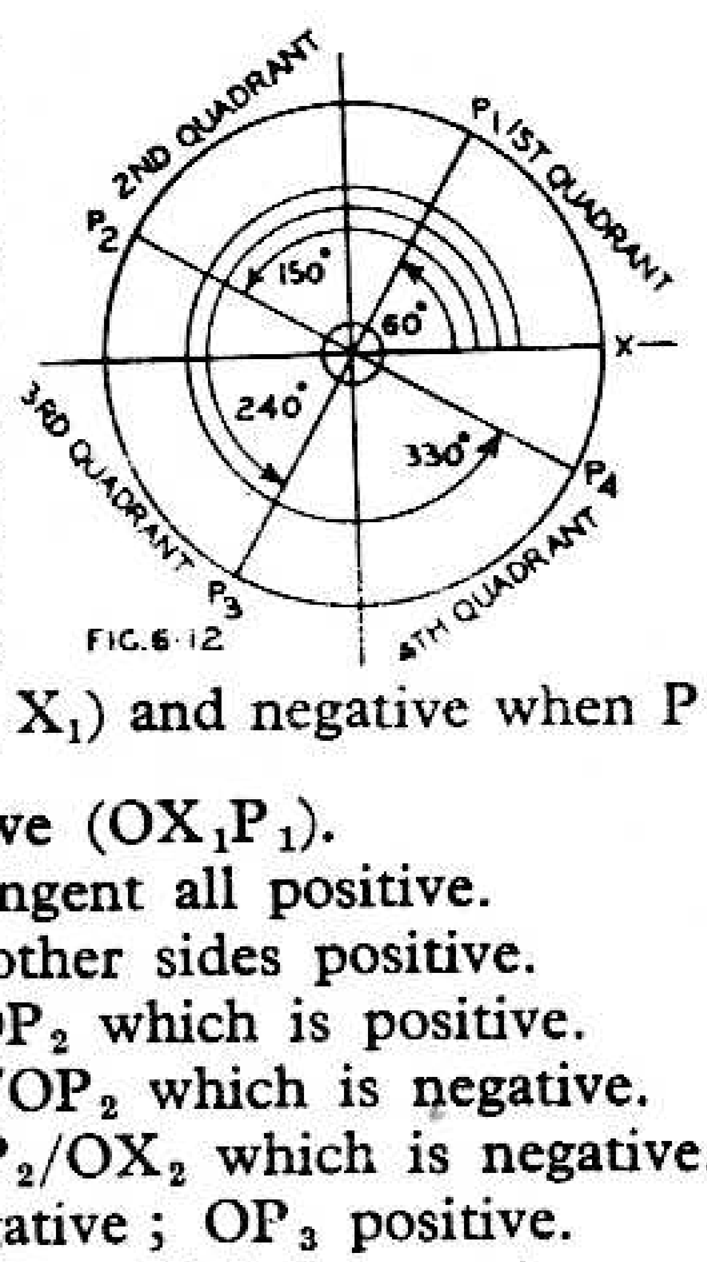


FIG. 6·12

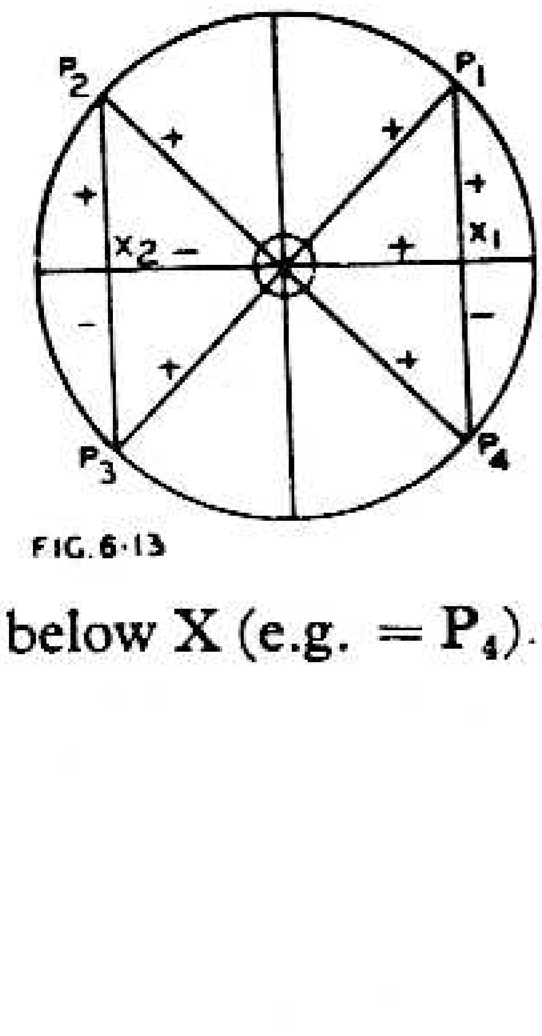


FIG. 6·13

In trigonometry it is also necessary to define the polarity of the three sides of the triangles from which we derive the sine, cosine and tangent. The hypotenuse (OP_1) is always positive (see Fig. 6.13). The base (OX_1) is positive when X is to the right hand side of O, and negative when X is to the left of O (e.g. OX_2). The perpendicular is positive when P is above X (e.g. $P_1 X_1$) and negative when P is below X (e.g. $= P_4$).

1st Quadrant: All sides positive (OX_1P_1).
Sine, cosine, tangent all positive.

2nd Quadrant: OX_2 negative, other sides positive.
Sine $= X_2P_2/OP_2$ which is positive.
Cosine $= OX_2/OP_2$ which is negative.
Tangent $= X_2P_2/OX_2$ which is negative.

3rd Quadrant: OX_2, X_2P_3 negative; OP_3 positive.
Sine $= X_2P_3/OP_3$ which is negative.
Cosine $= OX_2/OP_3$ which is negative.
Tangent $= X_2P_3/OX_2$ which is positive.

4th Quadrant: OX_1, OP_4 positive, X_1P_4 negative.
Sine $= X_1P_4/OP_4$ which is negative.
Cosine $= OX_1/OP_4$ which is positive.
Tangent $= X_1P_4/OX_1$ which is negative.

A convenient method for determining graphically the value of the sine, cosine and tangent of an angle is shown in Fig. 6.14, where the circle is drawn with radius 1 (to any convenient scale, say 1 inch). To the same scale, PX gives the value of the sine, OX the cosine and AT the tangent.

" Inverse " functions

If $\sin \theta = n$, we may describe θ as the angle whose sine is n. This is conventionally written in the form

$$\theta = \sin^{-1} n,$$

where the " $\sin^{-1}$ " is to be regarded purely as an abbreviation for " the angle whose sine is."

The same system is used with all trigonometrical functions—$\cos^{-1}$, $\tan^{-1}$, cosec^{-1}, $\sec^{-1}$, $\cot^{-1}$.

These are occasionally written as arc sin, arc cos, arc tan, etc.

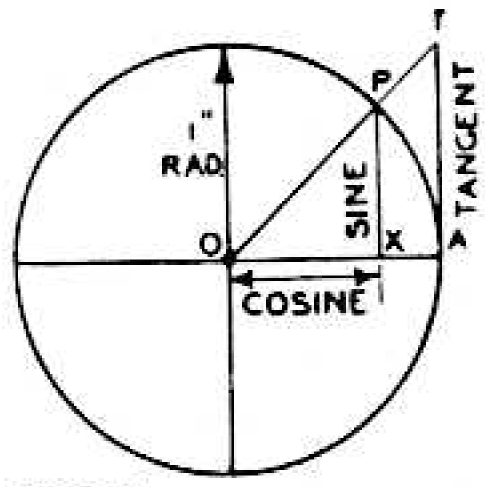


FIG. 6·14

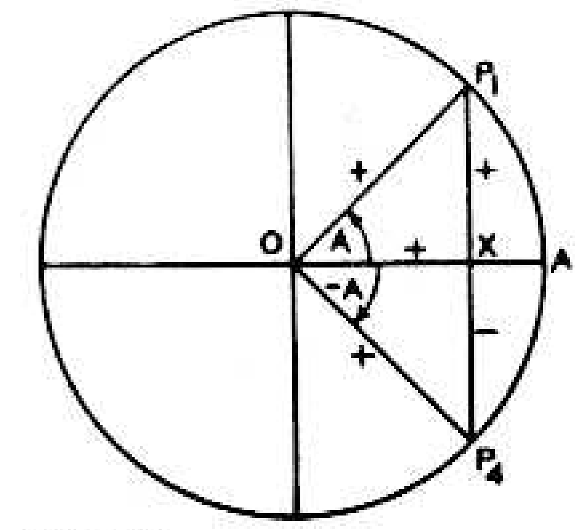


FIG. 6·15

Summary of trigonometrical relationships

1. $\text{cosec}\ A = 1/\sin A = \cot A/\cos A$ (69)
   $\sec A = 1/\cos A = \tan A/\sin A$ (70)
   $\cot A = 1/\tan A = \cos A/\sin A$ (71)

2. $\tan A = (\sin A)/(\cos A) = \sin A \sec A$ (72)

3. $\sin A = \cos(90° - A) = \sin(180° - A)$ (73)
   $\cos A = \sin(90° - A) = -\cos(180° - A)$ (74)
   $\tan A = \cot(90° - A) = -\tan(180° - A)$ (75)
   $\text{cosec}\ A = \sec(90° - A) = \text{cosec}\,(180° - A)$ (76)
   $\sec A = \text{cosec}\,(90° - A) = -\sec(180° - A)$ (77)
   $\cot A = \tan(90° - A) = -\cot(180° - A)$ (78)

4. $\sin^2 A + \cos^2 A = 1\ ;\ \sec^2 A - \tan^2 A = 1$ (79)
   (Note : $\sin^2 A$ is the square of $\sin A$)

$$\sin A = \pm\sqrt{1 - \cos^2 A} = \pm\frac{\tan A}{\sqrt{1 + \tan^2 A}} = \pm\frac{1}{\sqrt{1 + \cot^2 A}} \quad (80)$$

$$\cos A = \pm\sqrt{1 - \sin^2 A} = \pm\frac{1}{\sqrt{1 + \tan^2 A}} = \pm\frac{\cot A}{\sqrt{1 + \cot^2 A}} \quad (81)$$

   the choice of signs being determined by the quadrant.

5. $\sec^2 A = 1/\cos^2 A = 1 + \tan^2 A$ (82)
   $\text{cosec}^2 A = 1/\sin^2 A = 1 + \cot^2 A$ (83)

6. $\sin(A + B) = \sin A \cos B + \cos A \sin B$ (84)
   $\cos(A + B) = \cos A \cos B - \sin A \sin B$ (85)

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \quad (86)$$

7. $\sin(A - B) = \sin A \cos B - \cos A \sin B$ (87)
   $\cos(A - B) = \cos A \cos B + \sin A \sin B$ (88)

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \quad (89)$$

8. Negative angles (Fig. 6.15)
   $\sin(-A) = XP_4/OP_4 = -(XP_1/OP_1) = -\sin A$ (90)
   $\cos(-A) = OX/OP_4 = OX/OP_1 = \cos A$ (91)
   $\tan(-A) = XP_4/OX = -(XP_1/OX) = -\tan A$ (92)
   $\text{cosec}\,(-A) = OP_4/XP_4 = -(OP_1/XP_1) = -\text{cosec}\ A$ (93)
   $\sec(-A) = OP_4/OX = OP_1/OX = \sec A$ (94)
   $\cot(-A) = OX/XP_4 = -(OX/XP_1) = -\cot A$ (95)

9. $$\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1 + \tan^2 A} \quad (96)$$

$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1 \quad (97)$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}\ ;\ \cot 2A = \frac{\cot^2 A - 1}{2\cot A} \quad (98)$$

10. $\sin \tfrac{1}{2}A = \pm\sqrt{\tfrac{1}{2}(1 - \cos A)} = \pm\left(\tfrac{1}{2}\sqrt{1 + \sin A} - \tfrac{1}{2}\sqrt{1 - \sin A}\right)$ (99)
    $\cos \tfrac{1}{2}A = \pm\sqrt{\tfrac{1}{2}(1 + \cos A)} = \pm\left(\tfrac{1}{2}\sqrt{1 + \sin A} + \tfrac{1}{2}\sqrt{1 - \sin A}\right)$ (100)

$$\tan \tfrac{1}{2}A = \frac{1 - \cos A}{\sin A} = \frac{\sin A}{1 + \cos A} \quad (101)$$

11. $\sin 3A = 3\sin A - 4\sin^3 A$ $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$ (102)
    $\cos 3A = 4\cos^3 A - 3\cos A$

12. $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$ (103)
    $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$ (104)

13. $\sin^3 A = \frac{1}{4}(3\sin A - \sin 3A)$ (105)
    $\cos^3 A = \frac{1}{4}(\cos 3A + 3\cos A)$ (106)

14. Approximations for small angles :
    where A is measured in radians
    $\sin A \approx A - (A^3)/6$ error for 30° is 0.06% (107)
    $\cos A \approx 1 - (A^2)/2$ error for 30° is 0.35% (108)
    $\tan A \approx A + (A^3)/3$ error for 30° is 1.03% (109)
    and for very small angles $\sin A \approx A$ (110)
    $\cos A \approx 1$ (111)
    $\tan A \approx A.$ (112)

    See also Sect. 2(xx).

15. $\sin A = \dfrac{\epsilon^{jA} - \epsilon^{-jA}}{2j}$ [Sect. 6, eqn. (19)] (113)

    $\cos A = \dfrac{\epsilon^{jA} + \epsilon^{-jA}}{2}$ [Sect. 6, eqn. (20)] (114)

16. versine $A = 1 - \cos A = 2\sin^2(A/2)$ (115)

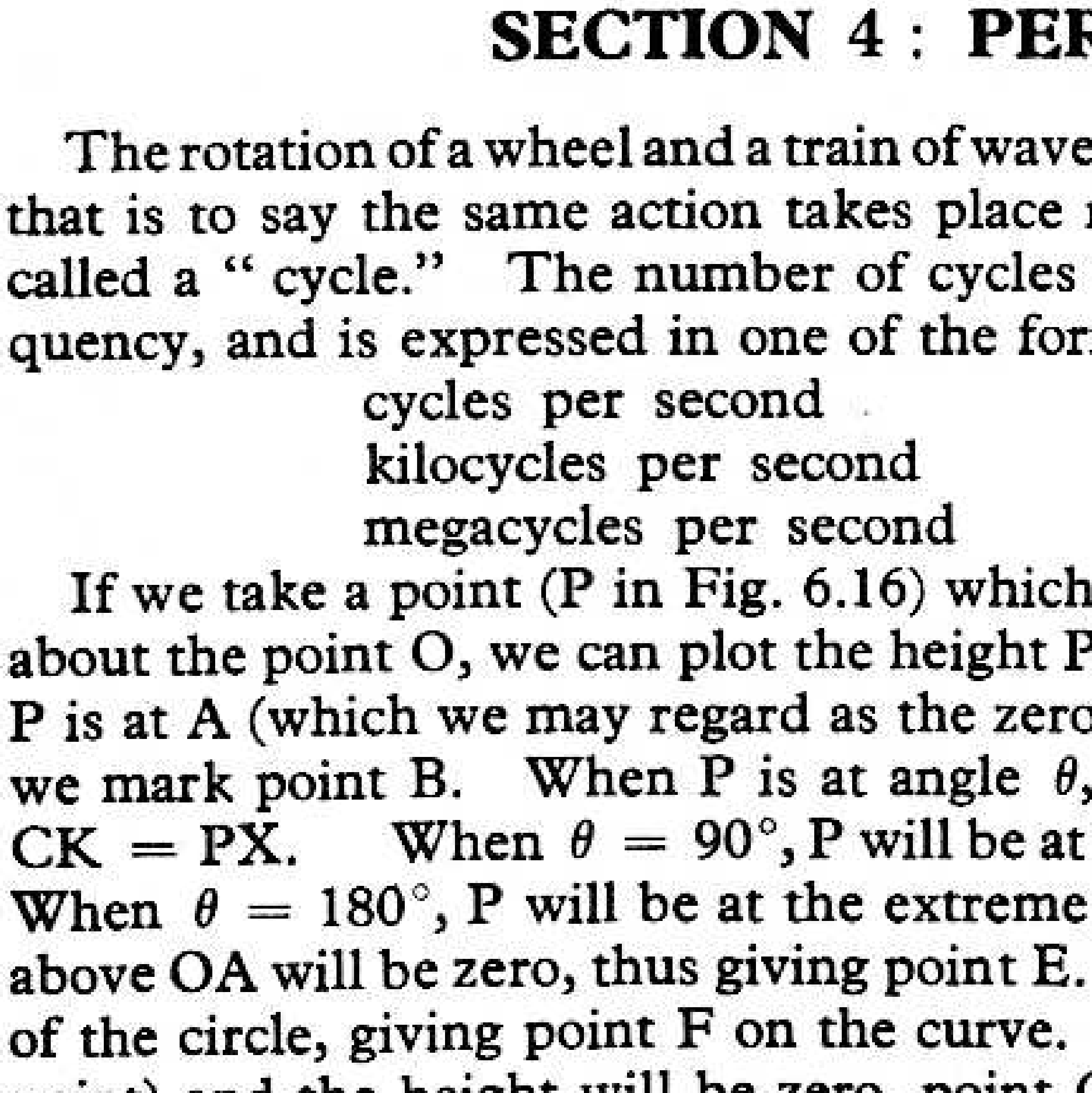
SECTION 4 : PERIODIC PHENOMENA

The rotation of a wheel and a train of waves are two examples of periodic phenomena, that is to say the same action takes place repeatedly, each of such phenomena being called a " cycle." The number of cycles which occur in 1 second is called the frequency, and is expressed in one of the forms

cycles per second	c/s
kilocycles per second	Kc/s
megacycles per second	Mc/s.

If we take a point (P in Fig. 6.16) which is rotating with uniform angular velocity* about the point O, we can plot the height PX against the angle of rotation (θ). When P is at A (which we may regard as the zero point, since here $\theta = 0$), PX = O, and we mark point B. When P is at angle θ, the perpendicular PX gives point C, and CK = PX. When $\theta = 90°$, P will be at the top of the circle, giving D on the curve. When $\theta = 180°$, P will be at the extreme left hand side of the circle, and the height above OA will be zero, thus giving point E. When $\theta = 270°$, P will be at the bottom of the circle, giving point F on the curve. Lastly, P will return to point A (the zero point) and the height will be zero, point G. If the process is continued, the curve (GH etc.) will repeat the shape of the first cycle (BD etc.) and so on indefinitely. Thus BCDEFG represents one cycle.

FIG. 6.16



The length PX = $r\sin\theta$, so that its projected height CK is proportional to the sine of θ. The curve is therefore called a " sine curve " or " sine wave." A cosine curve has exactly the same shape, except that it begins at D (since $\cos 0° = 1$) and the cycle ends at H.

The motion of the point X, as it oscillates about O between the extremes A and A′ is called **Simple Harmonic Motion.**

Angular velocity (usually represented by the small Greek letter omega—ω) is the number of radians per second through which the point P travels. In each revolution

*i.e. uniform rate of rotation.

(360°) it will pass through 2π radians, and if it makes f revolutions per second, then the angular velocity will be

$$\omega = 2\pi f \text{ radians per second}$$
$$\text{where } f = \text{frequency in cycles per second.}$$

In most mathematical work it is more convenient to write ω than to write $2\pi f$.

SECTION 5 : GRAPHICAL REPRESENTATION AND *j* NOTATION

(i) Graphs (ii) Finding the equation to a curve (iii) Three variables (iv) Vectors and j notation.

(i) Graphs

Graphs are a convenient representation of the relationships between functions. For example, Ohm's Law

$$E = RI$$

may be represented by a graph (Fig. 6.17) in which I is plotted horizontally (on the X axis) and E vertically (on the Y axis) for a constant value of R.

Any point P on the plotted " curve " has its position fixed by coordinates. The horizontal, or $x-$coordinate (OQ) is called the abscissa, while the vertical, or $y-$coordinate (QP) is called the ordinate. The position of the point is written as a, b, thus indicating that $x = a$ and $y = b$, where $a =$ OQ and $b =$ QP.

Any function of the form

$$y = mx + n \quad (m \text{ and } n \text{ being constants})$$

is a linear equation, since the plotted curve is a straight line. It only passes through the origin (O) if $n = 0$

In the general case, the axes extend in both directions about the origin (Fig. 6.18) forming four quadrants and allowing for negative values of both x and y. These are known as " Cartesian Coordinates."

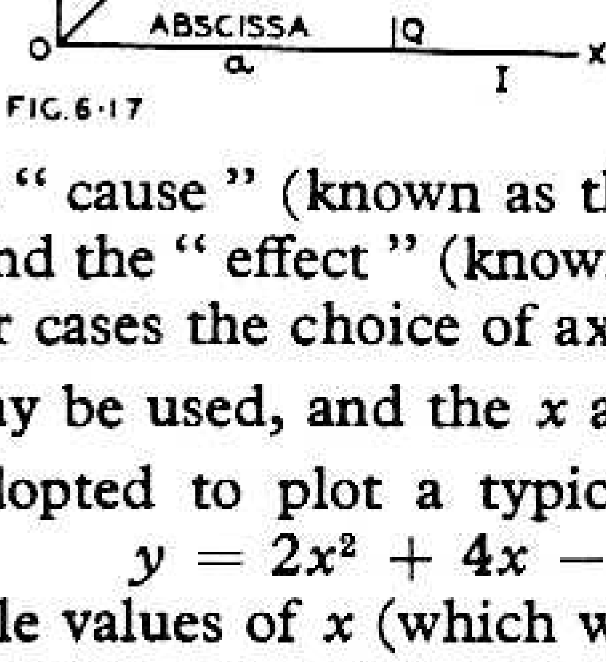


FIG. 6·17

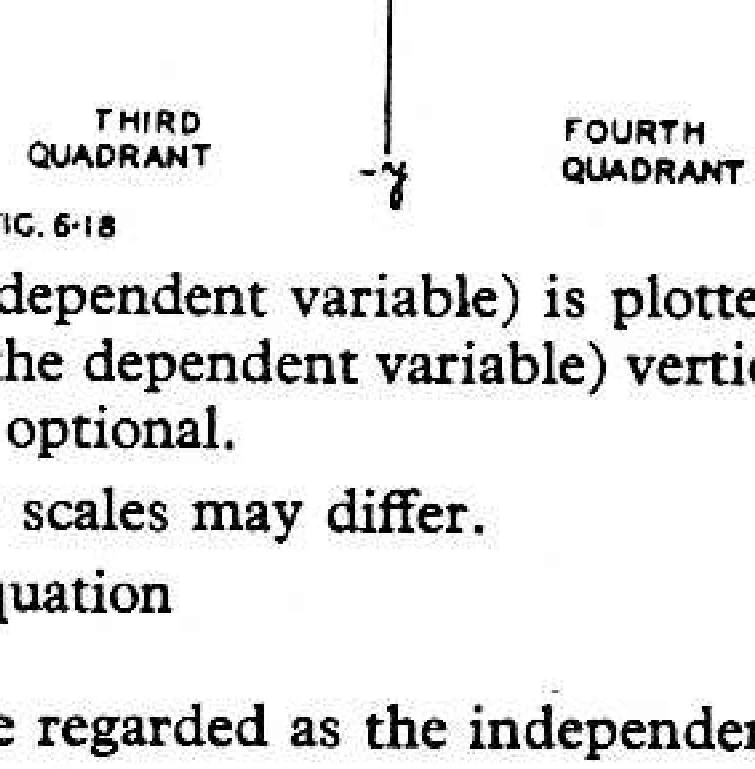


FIG. 6·18

When the two variable quantities in an equation can be separated into " cause " and " effect," the " cause " (known as the independent variable) is plotted horizontally on the x axis, and the " effect " (known as the dependent variable) vertically on the y axis. In other cases the choice of axes is optional.

Any convenient scales may be used, and the x and y scales may differ.

The procedure to be adopted to plot a typical equation

$$y = 2x^2 + 4x - 5$$

is as follows. Select suitable values of x (which will be regarded as the independent variable) and calculate the value of y for each :

$$\begin{array}{ll}
x = 3 & y = 18 + 12 - 5 = +25 \\
x = 2 & y = 8 + 8 - 5 = +11 \\
x = 1 & y = 2 + 4 - 5 = +1 \\
x = 0 & y = 0 + 0 - 5 = -5 \\
x = -1 & y = 2 - 4 - 5 = -7 \\
x = -2 & y = 8 - 8 - 5 = -5 \\
x = -3 & y = 18 - 12 - 5 = +1
\end{array}$$

Then plot these points, as in Fig. 6.19A and draw a smooth curve through them.

The tangent at any point (e.g. P in Fig. 6.19A) is a straight line which is drawn so as to touch the curve at the point. The slope of the tangent, which is the same as that of the curve at the point, is defined as the tangent of the angle θ which it makes with the X axis. Between points B and C on the curve, θ is positive, therefore tan θ is positive and the slope is called positive. Between points B and A the angle θ is negative, and the slope negative. At point B, $\theta = 0$ and the slope is zero. It is important to remember that the curve normally extends in both directions indefinitely unless it has limits, or turns back on itself. It is therefore advisable, when plotting an unknown function, to take very large positive and negative values of x and calculate the corresponding values of y, even though the points cannot be plotted on the graph paper. This will indicate the general trend of the curve beyond the limits of the graph paper.

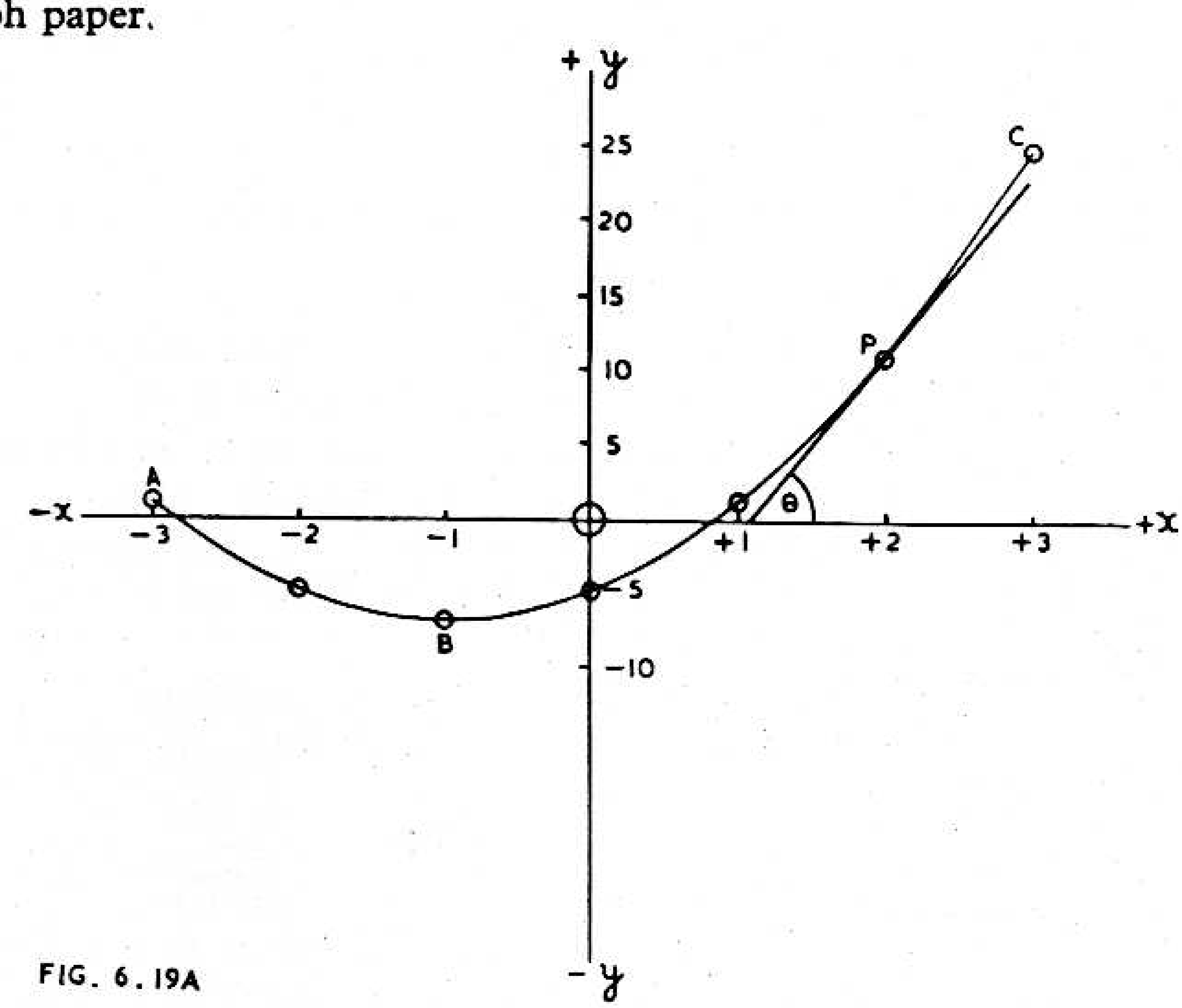


FIG. 6.19A

The equations of some common curves are :

Straight line through origin	$y = mx$	(1)
Straight line not through origin	$y = mx + n$	(2)
Circle with centre at origin	$y^2 = r^2 - x^2$	(3)
Circle with centre at (h, k)	$(x - h)^2 + (y - k)^2 = r^2$	(4)
General equation of circle	$x^2 + y^2 + dx + ey + f = 0$	(5)
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	(6)
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	(7)
Parabola (origin at vertex)	$y^2 = 2px$	(8)
Focus is at	$x = p/2, y = 0$	(9)

Areas and average heights

The average height of a curve (i.e. the average length of the ordinates) may be determined by dividing the area beneath the curve into strips of equal width, and then using either the Trapezoid Rule or Simpson's Rule [see Sect. 3 eqn. (54)] to determine the area, and dividing the area by the length (abscissa).

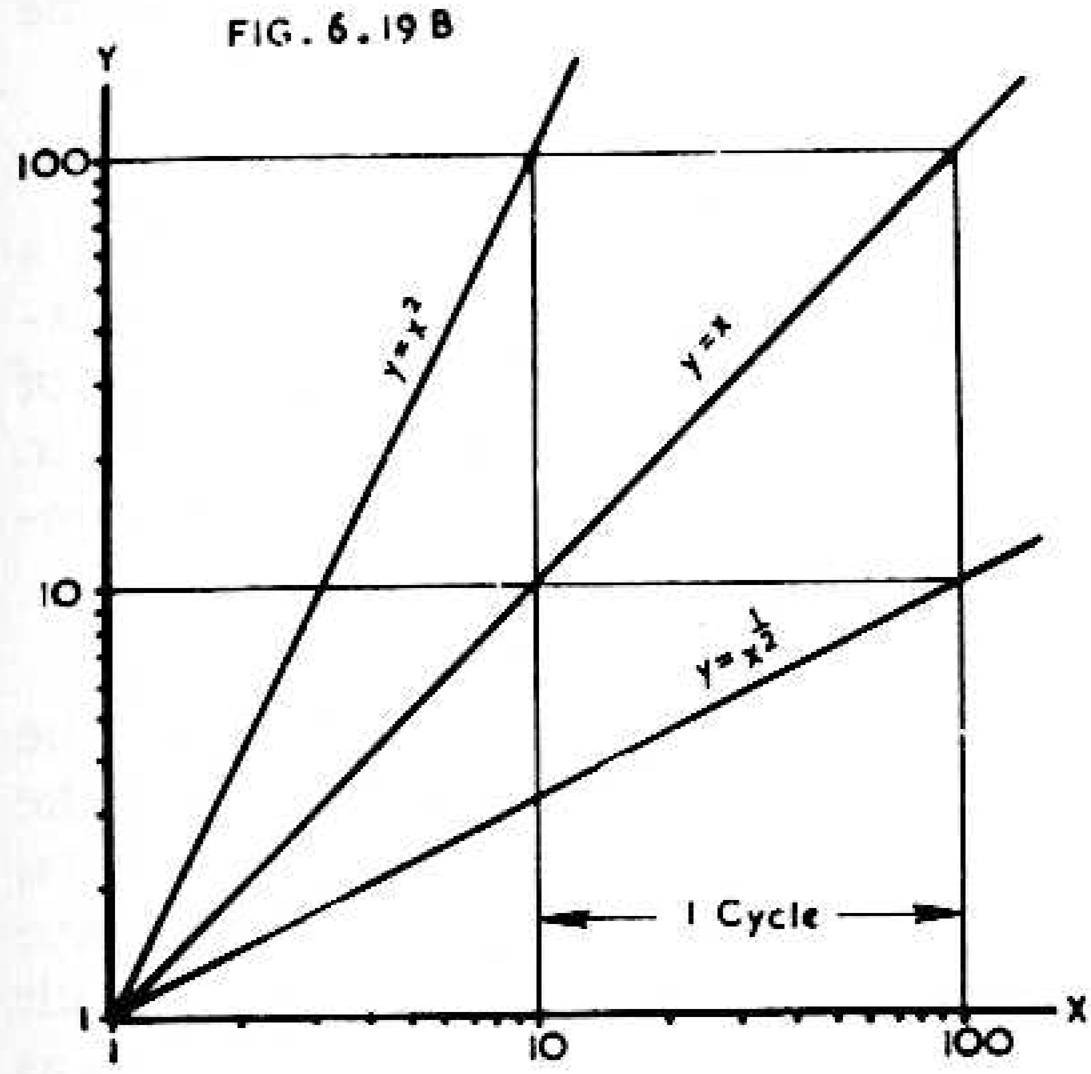
Logarithmic paper

Logarithmic ruled paper is frequently used in the plotting of curves, particularly when the x or y coordinates cover a range of 100 : 1 or more. Single cycle* paper accommodates a range of 10 : 1, and may be drawn by hand, using the whole of the C scale, or half the B scale on a slide rule. Two " cycle " log paper accommodates a range of 100 : 1, and may be drawn with the whole B scale on a slide rule. Each

*Strictly this should be called single decade.

of the " cycles " has the same linear length. Additional " cycles " may be added as desired (for examples see a.v.c. characteristics in Chapter 27).

FIG. 6.19B



Occasionally it is desirable to use log. log. ruled paper ; this is ruled logarithmically on both X and Y axes. An important feature of this form of representation is that a curve of the type

$$y = ax^n$$

is shown as a straight line with a slope of n where the slope is measured as the number of " cycles " on the Y axis per " cycle " on the X axis (see Fig. 6.19B).

In plotting readings (say for a valve I_b, E_b characteristic) which are likely to follow a power law, it is often helpful to use log. log. paper. A straight line indicates a true power law, and its slope gives the value of the exponential—usually not an integer. A slightly curving line indicates a close approach to a power law, and a tangent or chord may be drawn to give the slope at a point or the average over a region.

(ii) Finding the equation to a curve

A method which may be used to find the equation to a curve is given by K. R. Sturley in his book " Radio Receiver Design—Part 1 " (Chapman and Hall, London, 1943) pages 419-421.

(iii) Three variables

The plate current of a triode is given approximately by

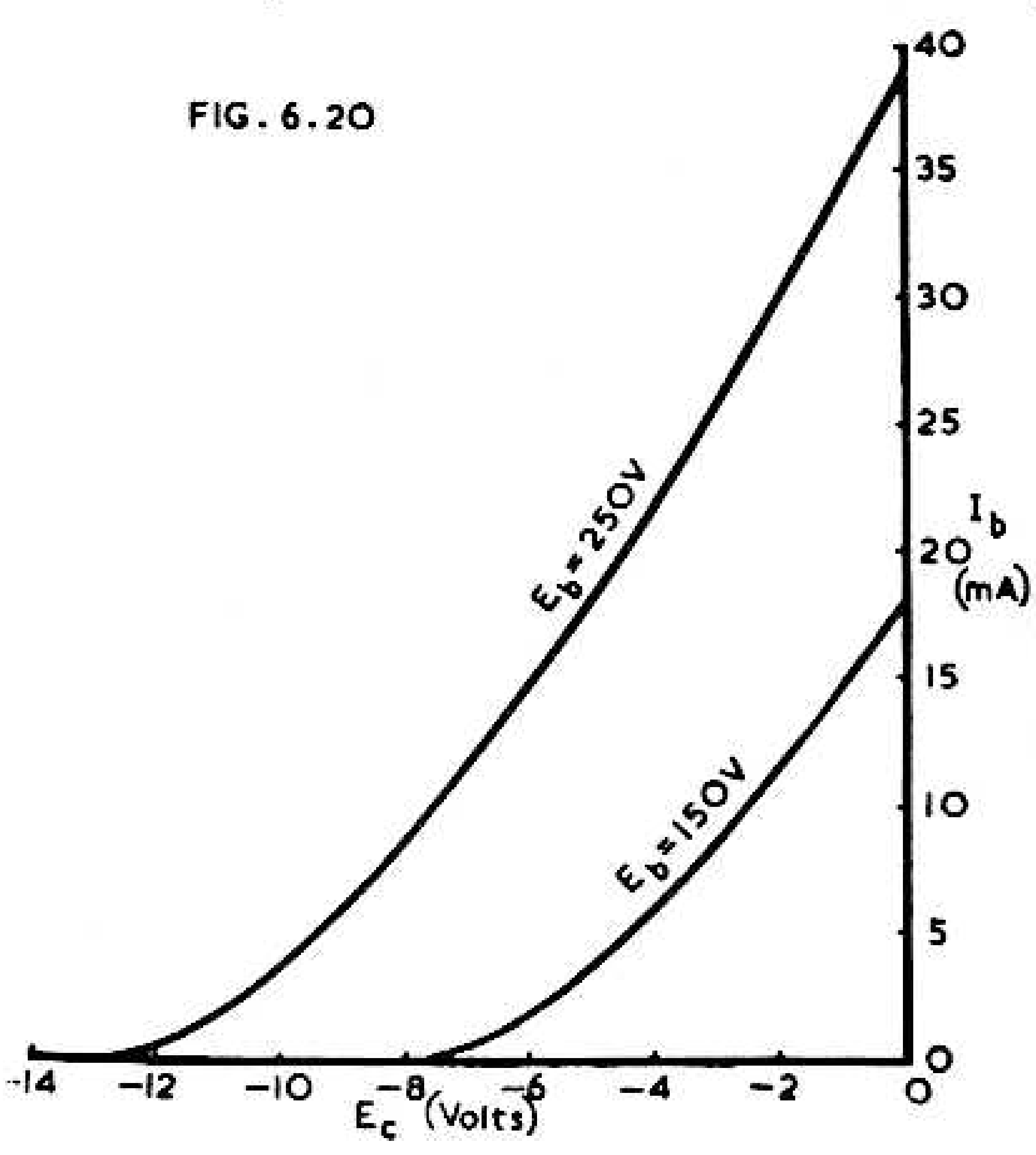
$$I_b = K(\mu E_c + E_b)^{3/2}.$$

In one typical case $K = 10 \times 10^{-6}$ mhos (the perveance)

and $\mu = 20$.

Here there are three variables, E_c, E_b and I_b. We can select a suitable value of E_b, calculate the curve, and then repeat the process for other values of E_b. Here E_b is called the **parameter.**

FIG. 6.20



If $E_b = 250$ volts, the plate current in milliamperes will be

$$I_b = 10^{-2} (20 \times E_g + 250)^{3/2} \text{ mA}$$

If $E_c = 0, I_b = 10^{-2} (0 + 250)^{3/2} = 10^{-2} (250)^{3/2} \approx 39$ mA

$E_c = -4, I_b = 10^{-2} (-80 + 250)^{3/2} = 10^{-2} (170)^{3/2} \approx 22$ mA

$E_c = -8, I_b = 10^{-2} (-160 + 250)^{3/2} = 10^{-2} (90)^{3/2} \approx 8.5$ mA

$E_c = -12, I_b = 10^{-2} (-240 + 250)^{3/2} = 10^{-2} (10)^{3/2} \approx 0.3$ mA

These points have been plotted in Fig. 6.20, and a curve has been drawn through them marked $E_b = 250$ V.

Similarly, for $E_b = 150$ V,

If $E_c = 0, I_b = 10^{-2} (0 + 150)^{3/2} = 10^{-2} (150)^{3/2} \approx 18$ mA

$E_c = -2, I_b = 10^{-2} (-40 + 150)^{3/2} = 10^{-2} (110)^{3/2} \approx 11.5$ mA

$E_c = -4, I_b = 10^{-2} (-80 + 150)^{3/2} = 10^{-2} (70)^{3/2} \approx 5.8$ mA

$E_c = -6, I_b = 10^{-2} (-120 + 150)^{3/2} = 10^{-2} (30)^{3/2} \approx 1.6$ mA

Cut off occurs at $E_c = -(150/20) = -7.5$ volts.

These points have been plotted and a smooth curve drawn through them. Similar curves could be drawn for any other plate voltage, thus forming a " family " of curves. This is actually a three-dimensional graphical diagram reduced to a form suitable for a flat surface.

(iv) Vectors and *j* notation

Any physical quantity which possesses both magnitude and direction is called a vector. Vectors may be represented on paper by means of straight lines with arrow-heads. The length of the line indicates (to some arbitrary scale) the magnitude of the quantity, and the direction of the line and arrow-head indicates the direction in which the vector is operating. The position of the line on the paper is of no consequence.

Addition of vectors

Vectors may be added by drawing them in tandem, and taking the resultant from the beginning of the first one to the end of the last one. Fig. 6.21A shows two vectors, **a** and **b**, which are added together to give the resultant **c**. Exactly the same result is obtained by placing **a** and **b** together, as in Fig. 6.21B, completing the parallelogram, and taking **c** as the diagonal. Vectors are generally printed in bold face type, to distinguish them from **scalar** values, which have no direction, although they have magnitude and sign (i.e. positive or negative).

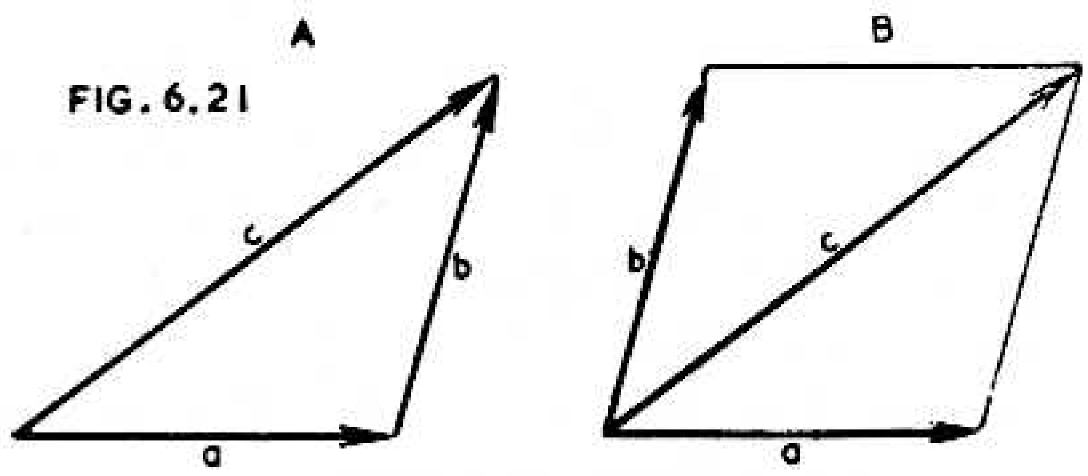


FIG. 6.21

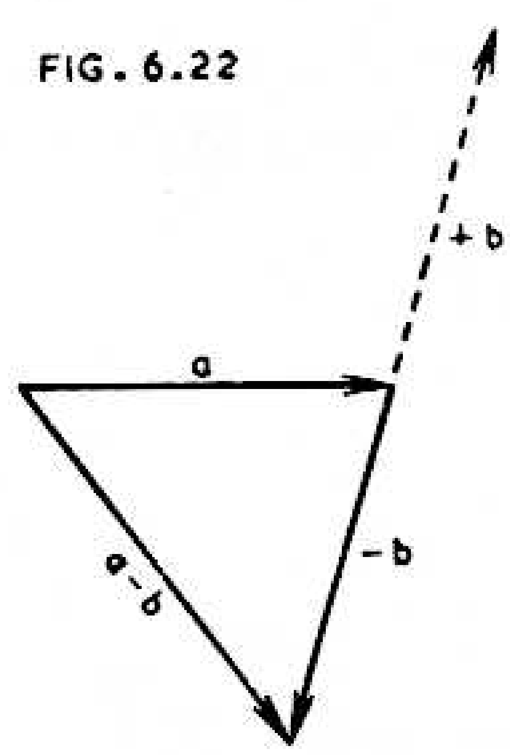


FIG. 6.22

Vector negative sign

A vector (– **a**) has the same magnitude as a vector (**a**) but its direction is reversed.

Subtraction of vectors

The vector to be subtracted is reversed in direction, and then the vectors are added.

In Fig. 6.22, to find **a** – **b**, the direction of **b** is reversed to give (– **b**) and then **a** and (– **b**) are added to give the resultant (**a** – **b**).

Multiplication of a vector by a number (n)

The resultant vector has the same direction, but its length is increased n times

e.g. $\mathbf{a} \times n = n\mathbf{a}$

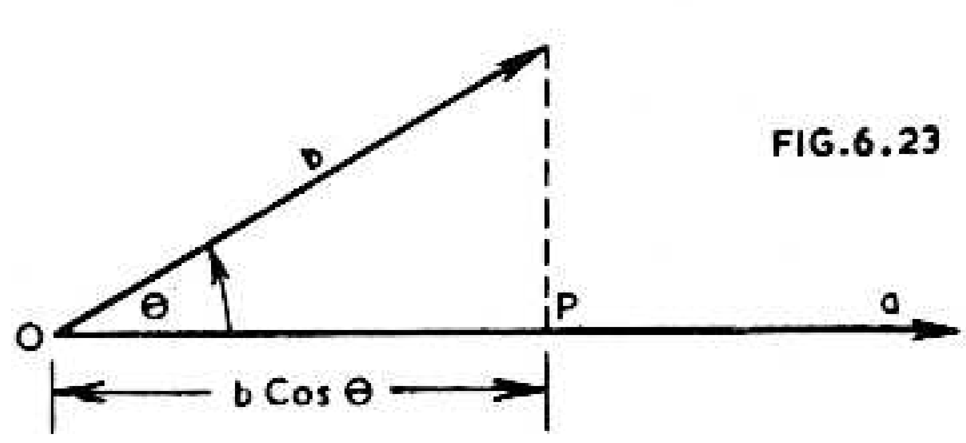


FIG. 6.23

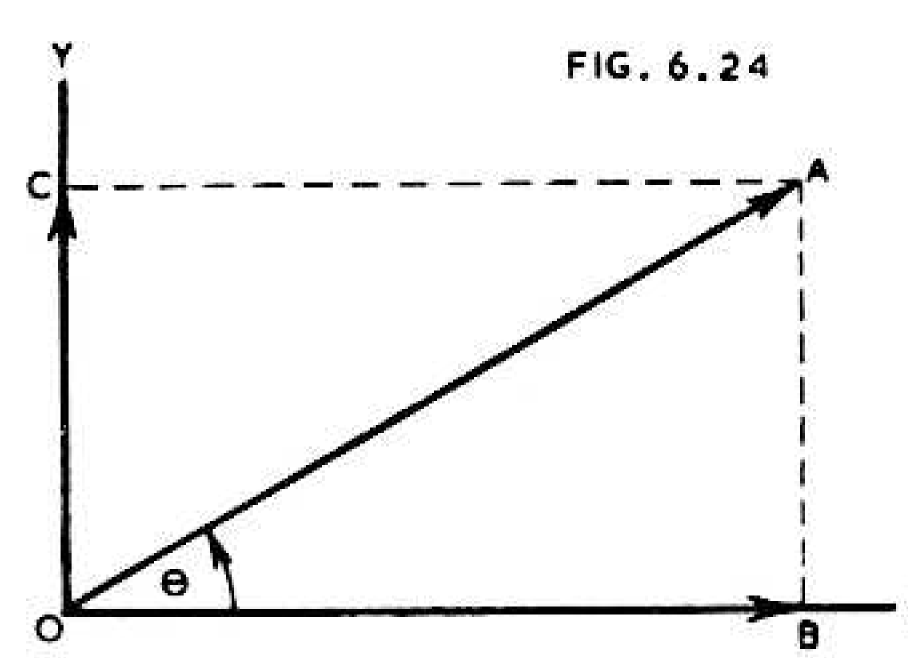


FIG. 6.24

The scalar product of two vectors

The scalar product of two vectors (**a** and **b** in Fig. 6.23) is $ab \cos \theta$, where θ is the angle between them. This may be written

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \theta$$

where **a** . **b** indicates the multiplication of two vectors. From Fig. 6.23 it will be seen that the scalar product is the product of the magnitude of one vector and the " projection " of the other on it.

Components of a vector

Any vector can be resolved into two component vectors in any two desired directions. For example, in Fig. 6.21 the vector **c** can be resolved into the component vectors **a** and **b**. If the component vectors are at right angles to one another they are called rectangular components ; in such a case they are usually taken horizontally (along the X axis) and vertically (along the Y axis). The vector **OA** in Fig. 6.24 can be resolved into two rectangular components **OB** and **OC** where

$$|\mathbf{OB}| = |\mathbf{OA}| \cos \theta, \ |\mathbf{OC}| = |\mathbf{OA}| \sin \theta.$$

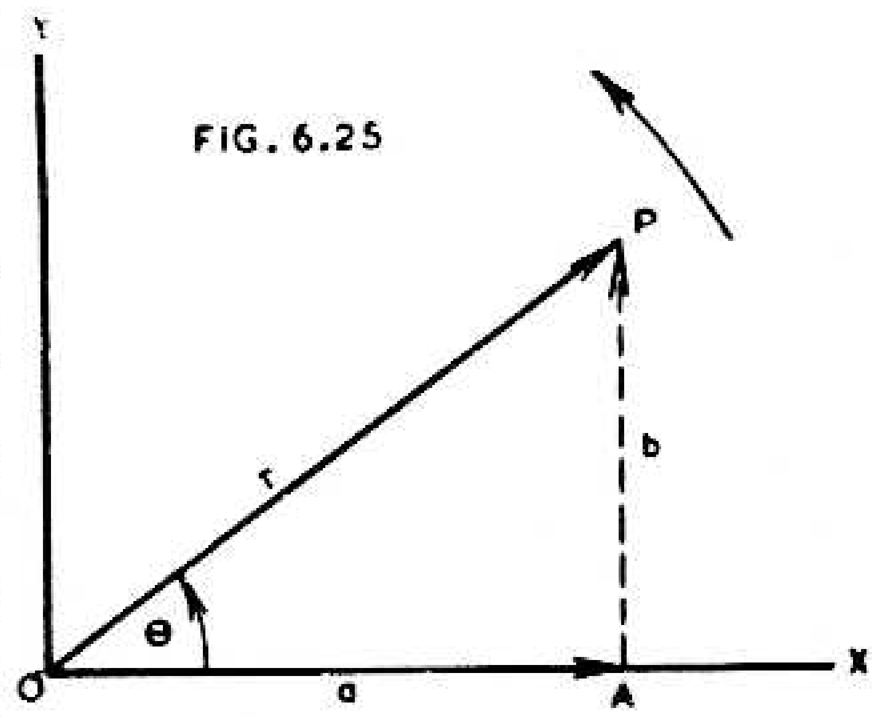
Polar coordinates

An alternative form of defining a vector **OP** is

$$\mathbf{OP} = r \angle \theta$$

where r is the magnitude of OP, and $\angle \theta$ indicates that there is an angle θ between it and OX (Fig. 6.25).

A graphical device has been described* for the conversion from complex to polar forms.



Rotating vectors** (*j* **notation)**

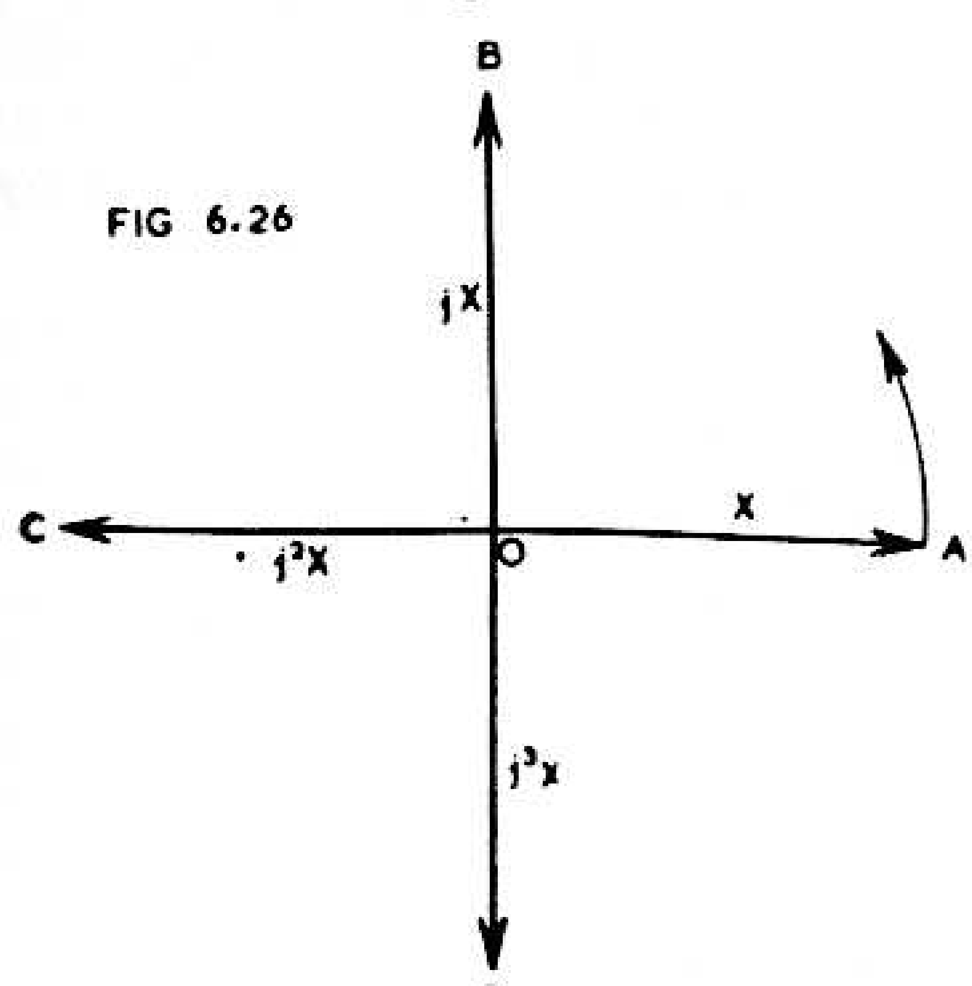
If a vector **X** (OA in Fig. 6.26) is rotated 90° in a positive direction to a position OB, the new vector is called j**X**, and j **is described as an " operator " which rotates a vector by 90° without changing its magnitude.**

If the vector j**X** (OB in Fig. 6.26) is operated upon by j, it will be rotated 90° to the position OC, where it is called j^2X, the j^2 indicating that it has been rotated $2 \times 90° = 180°$ from its original position OA.

If the vector j^2**X** (OC in Fig. 6.26) is operated upon by j, it will be rotated 90° to the position OD, where it is called j^3**X**, the j^3 indicating that it has been rotated $3 \times 90° = 270°$ from its original position OA.

If the vector j^3**X** (OD in Fig. 6.26) is operated upon by j, it will be rotated 90° to the position OA where it would be called j^4**X**, the j^4 indicating that it has been rotated $4 \times 90° = 360°$ from its original position OA.

There is an important deduction which is immediately obvious. j^2 **indicates a reversal of direction which is the same as a change of sign. The operator j^2 is therefore equivalent to multiplication by -1.**



If the operator j^2 is applied twice in succession, the result should be equivalent to multiplication by -1 twice in succession, i.e. $(-1) \times (-1) = +1$. This is so, because the operator j^4 brings the vector back to its original direction.

Since the operator j^2 is equivalent to multiplication by -1, we may deduce that the operator j is equivalent to multiplication by $\sqrt{-1}$, even though this in itself does not mean anything.

Operator	Equivalent to multiplication by
j	$\sqrt{-1}$
j^2	-1
j^3	$-\sqrt{-1}$
j^4	$+1$
$-j$	$-\sqrt{-1}$ (same as operator j^3)

*Snowdon, C. " A vector calculating device," Electronic Eng. 17.199 (Sept. 1944) 146.

**Also called " radius vectors." In pure mathematics " i " is used in place of " j."

Operator	Equivalent to multiplication by
$\frac{1}{j}$	$-\sqrt{-1}$ (same as operator $-j$)
$\frac{1}{-j} = -\frac{1}{j}$	$\sqrt{-1}$ (same as operator j)

See Chapter 4 Sects. 4(v), 5(v) and 6 for the application of the j notation to a.c. circuits.

The direction of any vector can be defined in terms of the j notation. In Fig. 6.25, OX represents the axis of reference, **OP** the vector, and θ the angle of rotation of OP from OX. The perpendicular PA may be drawn from P to OX and then, by the simple theory of vectors, **OP** is the sum of the vectors **OA** and **AP**. Using j notation, we may say that

$$\mathbf{OP} = a + jb$$

which means that the vector **OP** is the vector sum of a and b, where a is in the direction of the reference axis OX and b is rotated 90° from OX. In other words, a is the component of OP in the direction OX, and b is the component of OP in the direction OY. The values of a and b are given by

$$a = \text{OP} \cos \theta, \quad b = \text{OP} \sin \theta,$$

where $\theta = \tan^{-1} b/a$.

This is sometimes called the Argand Diagram.

In electrical a.c. circuit theory (Chapter 4) it is well known that the current through an inductance lags behind the applied voltage, while the current through a capacitance leads the voltage. This is most clearly shown by rotating vectors. It should be borne in mind that all the vectors are rotating at the same angular velocity—one revolution per cycle of the applied voltage—and that the pictorial representation is for any one instant. Fig. 6.27 shows the vector diagram for peak a.c. current (I) flowing through a resistance (R) and an inductance (L) in series. The current vector (**I**) may be placed in any convenient direction—say horizontally (Fig. 6.27) ; a solid arrow head is used to distinguish it from voltage vectors. The peak voltage drop (**RI**) in the resistance must be " in phase " with the current, and is so shown. The scale to which the voltage vectors are drawn has no connection with the scale to which the current (**I**) is drawn—in fact we are here only concerned with the direction of **I**. The peak voltage drop across L is $\omega \mathbf{LI} = \mathbf{X}_L\mathbf{I}$, and this vector is drawn vertically so as to lead the current **I** by 90°. The total peak voltage drop (**E**) is found by " completing the parallelogram of vectors " as previously. It will be seen that **I** lags behind **E** by the angle θ.

FIG. 6.27

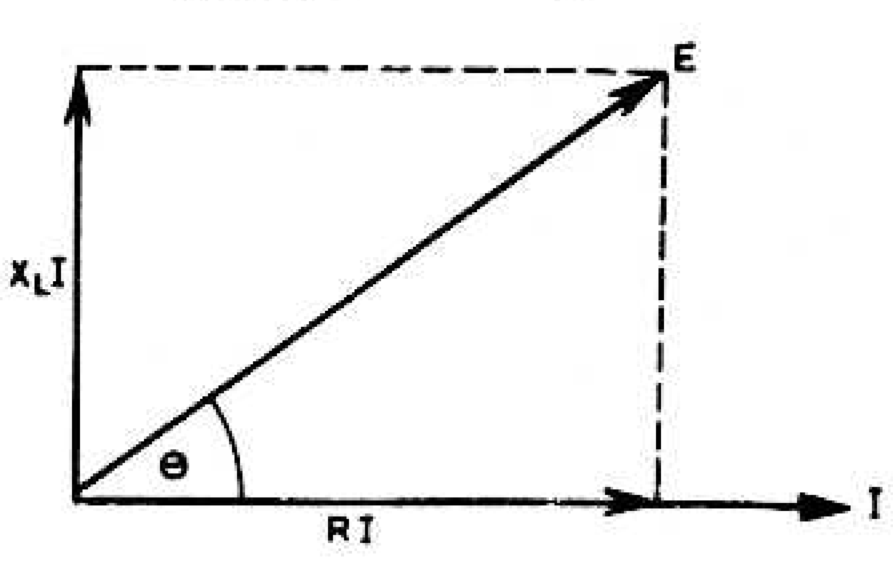
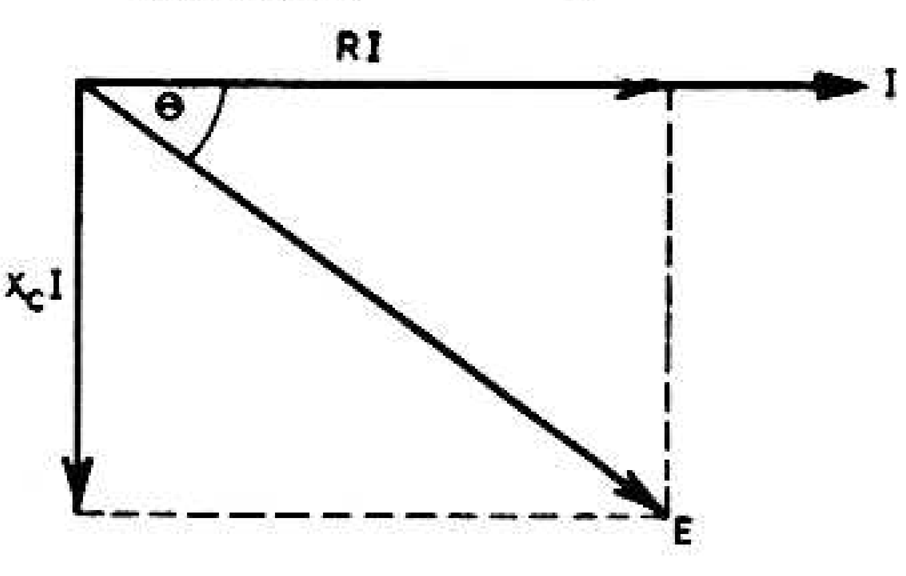


FIG. 6.28



A similar procedure applies with a capacitance instead of an inductance (Fig. 6.28) except that the capacitive reactance is X_c instead of X_L and the vector of peak voltage drop across C ($\mathbf{X}_c\mathbf{I}$) is drawn vertically downwards, since it must be opposite to $\mathbf{X}_L\mathbf{I}$. The current (**I**) here leads the voltage (**E**) by the angle θ.

SECTION 6: COMPLEX ALGEBRA AND DE MOIVRE'S THEOREM

(i) Complex algebra with rectangular coordinates (ii) Complex algebra with polar coordinates (iii) De Moivre's Theorem.

(i) Complex algebra with rectangular coordinates

Complex algebra should preferably be called " the algebra of complex quantities," and it is really quite simple to understand for anyone who has even a limited knowledge of mathematics.

FIG. 6.29

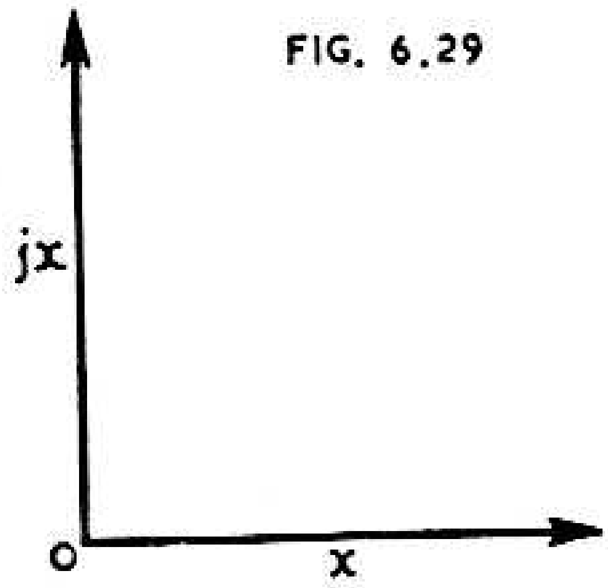
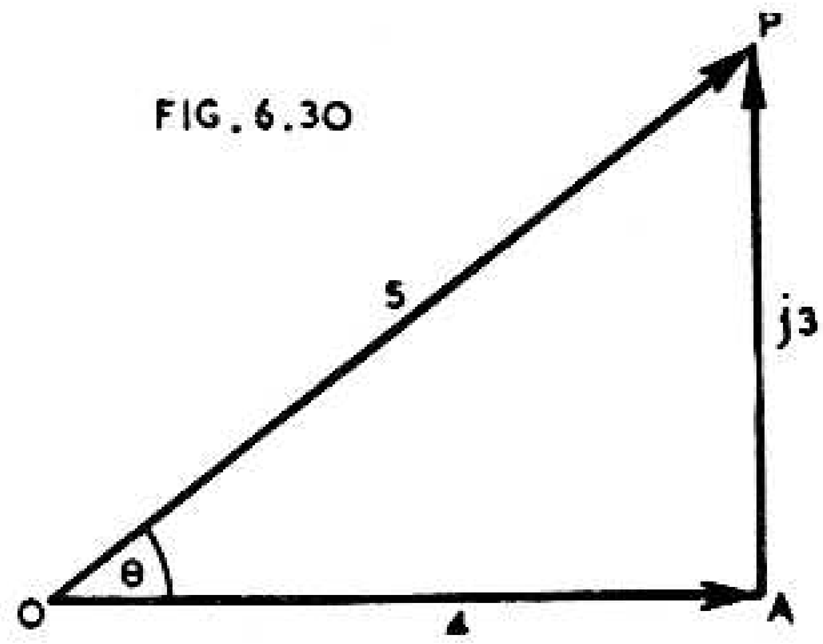


FIG. 6.30



It was explained in Section 5 that the letter j* in front of a vector indicated that it had been rotated 90° from the reference, or positive X, axis (Fig. 6.29). We can make use of this procedure to indicate the magnitude and direction of any vector. In Fig. 6.30 the vector **OP**, with magnitude 5, has been resolved into the rectangular components—

$$\mathbf{OA} = 4 \text{ in the } + X \text{ direction}$$
$$\text{and } \mathbf{AP} = 3 \text{ in the } jX \text{ direction } (90° \text{ to } X).$$

The X axis is taken horizontally through O, and is sometimes called the " real " axis, while the jX direction is called " imaginary." It is therefore possible to describe **OP** in both magnitude and direction by the expression

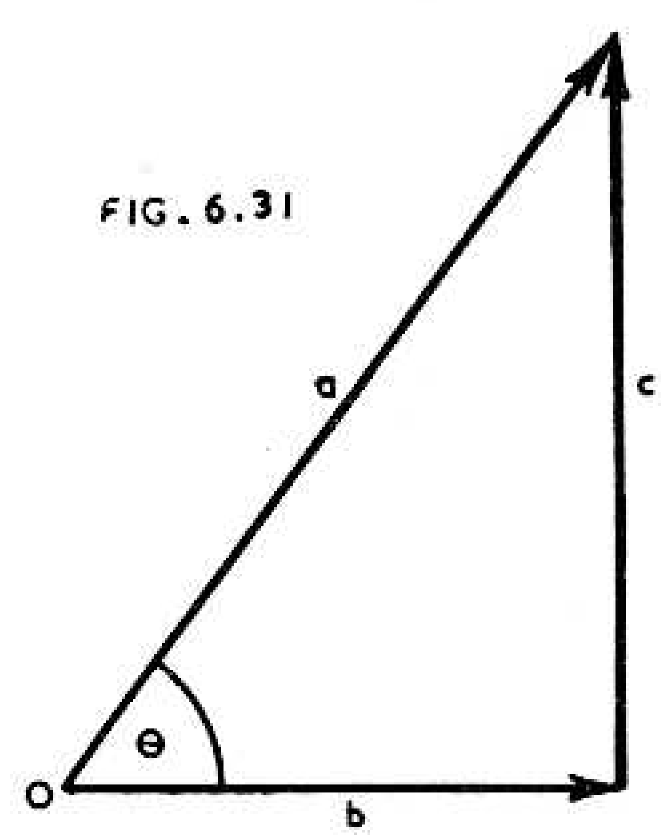
$$4 + j3.$$

Its magnitude is the vector sum of 4 and 3, which is $\sqrt{4^2 + 3^2} = \sqrt{25} = 5$. Its direction is given by the angle θ from the X axis, where $\cos\theta = 4/5$.

Any other vector, such as **a** in Fig. 6.31, can similarly be resolved into its components:

$$b = \mathbf{a} \cos\theta \text{ in the } X \text{ direction}$$
$$\text{and } c = \mathbf{a} \sin\theta \text{ in the } jX \text{ direction}$$

FIG. 6.31



and be written as

$$b + jc$$

where

$$b = \mathbf{a} \cos\theta$$
$$\text{and } c = \mathbf{a} \sin\theta.$$

The expression $b + jc$ is called a complex quantity, the word " complex " indicating that the addition is not to be made algebraically, but by vector addition.

This use of a complex quantity such as $b + jc$ is not limited to true vectors, but is found useful in many applications in electrical engineering. Its application to alternating currents is covered in Chapter 4, and the following treatment is a general introduction to the methods of handling complex quantities.

Modulus

The modulus is the magnitude of the original vector (**a** in Fig. 6.31) and is numerically equal to the square root of the sum of the squares of the magnitudes of the two components,

$$|a| = \sqrt{b^2 + c^2} \qquad (1)$$

Addition and subtraction of complex quantities

The primary rule is to place all " real " numbers together in one group and all " imaginary " numbers in another. These two groups must be kept entirely separate and distinct throughout. The final form should be

$$(\ldots) + j(\ldots).$$

*Or i in pure mathematics.

Example: Add $A + jB$, $C + jD$, $E - jF$.

Total is $A + C + E + jB + jD - jF$

$$= (A + C + E) + j(B + D - F).$$

Multiplication of complex quantities

The multiplication is performed in accordance with normal algebraic laws for real numbers, j^2 being treated as -1.

$$(x_1 + jy_1)(x_2 + jy_2) = x_1x_2 + jy_1x_2 + jy_2x_1 + j^2y_1y_2$$
$$= (x_1x_2 - y_1y_2) + j(x_1y_2 + x_2y_1) \quad (2)$$

Division of complex quantities

The denominator should be made a real number by multiplying both the numerator and the denominator by the conjugate of the denominator (i.e. the denominator with the opposite sign in front of the j term).

$$\frac{x_1 + jy_1}{x_2 + jy_2} = \frac{(x_1 + jy_1)(x_2 - jy_2)}{(x_2 + jy_2)(x_2 - jy_2)} = \frac{x_1x_2 + jy_1x_2 - jx_1y_2 + y_1y_2}{x_2^2 + y_2^2}$$
$$= \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}\right) - j\left(\frac{x_1y_2 - y_1x_2}{x_2^2 + y_2^2}\right) \quad (3)$$

Similarly

$$\frac{1}{x + jy} = \frac{(1)(x - jy)}{(x + jy)(x - jy)} = \left(\frac{x}{x^2 + y^2}\right) - j\left(\frac{y}{x^2 + y^2}\right) \quad (4)$$

Square root of complex quantities

To find the square root of $(a + jb)$, assume that the square root is $(x + jy)$ and then proceed to find x and y.

$$(x + jy)^2 = (x + jy)(x + jy) = x^2 - y^2 + 2jxy$$

Therefore $a + jb = (x^2 - y^2) + 2jxy = (x^2 - y^2) + j(2xy)$

Therefore $(x^2 - y^2) = a$

and $2xy = b.$

Modulus $= r = \sqrt{a^2 + b^2}$

Square of modulus of $(x + jy) = x^2 + y^2$

Therefore $x^2 + y^2 = r$

But $x^2 - y^2 = a$ (see above)

Therefore $x^2 = \frac{1}{2}(r + a)$ and $y^2 = \frac{1}{2}(r - a)$

Therefore $x = \pm\sqrt{\frac{1}{2}(r + a)}$ and $y = \pm\sqrt{\frac{1}{2}(r - a)}$ (5)

The signs should be checked to see which are applicable

(ii) Complex algebra with polar coordinates

In complex numbers we can write (Fig. 6.31)

$$a = b + jc$$

but $b = |a| \cos\theta$

and $c = |a| \sin\theta$

Therefore $a = |a|\cos\theta + j|a|\sin\theta$

Therefore $a = |a|(\cos\theta + j\sin\theta)$ (6)

Here $|a|$ is the magnitude of the vector and $(\cos\theta + j\sin\theta)$ may be called the **trigonometrical operator** which rotates the vector through the angle θ in a positive (counter-clockwise) direction from the x axis.

As explained in Sect. 5(iv), a vector may be defined by

$$r \angle \theta$$

where r is the magnitude and $\angle\theta$ the angle between the vector and the reference axis. It will therefore be seen that the trigonometrical operator $(\cos\theta + j\sin\theta)$ is effectively the same as $\angle\theta$ with polar coordinates.

i.e. $\angle\theta = \cos\theta + j\sin\theta$ (7)

Pure mathematical polar form

By the use of the Exponential Series we can express ϵ^x in the form (Sect. 2, eqn. 87)

$$\epsilon^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots$$

Putting $j\theta$ for x we obtain

$$\epsilon^{j\theta} = 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \ldots$$

$$= 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!} - \ldots$$

Grouping the j terms,

$$\epsilon^{j\theta} = (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots) + j(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots)$$

But $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots$ from eqn. (17) below

and $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$ from eqn. (18) below.

Therefore $\epsilon^{j\theta} = \cos\theta + j\sin\theta$ (8)

Thus the pure mathematical polar form is

$$|a|\,\epsilon^{j\theta}$$

and is called the Exponential Form.

The vector can thus be written in the various forms

$$a = b + jc \quad (9)$$

$$a = |a| \angle \theta \quad (10)$$

$$a = |a|\,\epsilon^{j\theta} \quad (11)$$

$$a = |a|(\cos\theta + j\sin\theta) \quad (12)$$

$$a = |a|\cos\theta + j|a|\sin\theta \quad (13)$$

A graphical method for converting from the complex form $(b + jc)$ to the polar form and vice versa has been described.* It will therefore be seen that there is a connection with the operator j:

j turns a vector through a right angle

$(\cos\theta + j\sin\theta)$ or $\epsilon^{j\theta}$ } turns a vector through an angle θ.

Addition and subtraction in polar form

Addition and subtraction may be done either graphically, or by expressing each in rectangular components and proceeding as for rectangular coordinates.

Multiplication of polar vectors

The product is found by multiplying their magnitudes and adding their angles.

Division of polar vectors

The quotient is found by dividing their magnitudes and subtracting the angle of the divisor from the angle of the dividend.

Square root of polar vectors

The root is found by taking the square root of the magnitude and half the angle.

(iii) De Moivre's Theorem

De Moivre's Theorem states that

$$(\cos\theta + j\sin\theta)^n = \cos n\theta + j\sin n\theta \quad (14)$$

where n may be positive or negative, fractional or integral.

It was explained in (ii) above that $(\cos\theta + j\sin\theta)$ may be regarded as a trigonometrical operator which rotates the vector through an angle θ. If this is applied twice in succession, the trigonometrical operator becomes

$$(\cos\theta + j\sin\theta)(\cos\theta + j\sin\theta) = (\cos\theta + j\sin\theta)^2$$

giving a total rotation of an angle 2θ.

Similarly if this is applied three times in succession, the trigonometrical operator becomes

$$(\cos\theta + j\sin\theta)(\cos\theta + j\sin\theta)(\cos\theta + j\sin\theta) = (\cos\theta + j\sin\theta)^3$$

giving a total rotation of an angle 3θ.

---

*Snowdon, C. "A vector calculating device" Electronic Eng. 17.199 (Sept. 1944) 146.

Thus, in the general case, the trigonometrical operator $(\cos\theta + j\sin\theta)^n$ gives a rotation of an angle $n\theta$ which, as explained above, is equivalent to a trigonometrical operator

$$(\cos n\theta + j\sin n\theta).$$

We have, in this way, proved De Moivre's Theorem for the case when n is a positive integer, and indicated the significance of the Theorem.

Application of De Moivre's Theorem :

1. To express $\cos n\theta$ and $\sin n\theta$ in terms of $\cos\theta$ and $\sin\theta$, where n is a positive integer.

   $\text{Cos}\, n\theta + j\sin n\theta = (\cos\theta + j\sin\theta)^n$

   $$= \cos^n\theta + nj\cos^{n-1}\theta.\sin\theta + \frac{n(n-1)}{2}j^2.\cos^{n-2}\theta.\sin^2\theta + \dots$$

   (as in Sect. 2, eqn. 83).

   Then equate the real and imaginary parts of the equation, giving

   $$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{2}\cos^{n-2}\theta.\sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{4!}\cos^{n-4}\theta.\sin^4\theta + \dots \quad (15)$$

   $$\sin n\theta = n\cos^{n-1}\theta.\sin\theta - \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta.\sin^3\theta + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!}\cos^{n-5}\theta.\sin^5\theta + \dots \quad (16)$$

2. To express $\cos\theta$ and $\sin\theta$ in terms of θ, write θ in the form $n(\theta/n)$ and expand the sine and cosine as in eqns. (15) and (16). When n becomes large, $\cos(\theta/n)$ may be taken as unity and $\sin(\theta/n)$ as (θ/n) itself. In the limit as n tends to infinity it can then be shown that

   $$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots \quad (17)$$

   $$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots \quad (18)$$

   where θ is expressed in radians.

3. To express $\sin\theta$ and $\cos\theta$ in terms of $\epsilon^{j\theta}$.

   By substituting $j\theta$ in place of x in the Exponential Series (Sect. 2, eqn. 87), and by using the relationship

   $$\epsilon^{j\theta} = \cos\theta + j\sin\theta$$

   and equations (17) and (18) we may obtain

   $$\sin\theta = \frac{\epsilon^{j\theta} - \epsilon^{-j\theta}}{2j} \quad (19)$$

   $$\text{and } \cos\theta = \frac{\epsilon^{j\theta} + \epsilon^{-j\theta}}{2} \quad (20)$$

SECTION 7: DIFFERENTIAL AND INTEGRAL CALCULUS

(i) Slope and rate of change (ii) Differentiation (iii) Integration (iv) Taylor's Series (v) Maclaurin's Series.

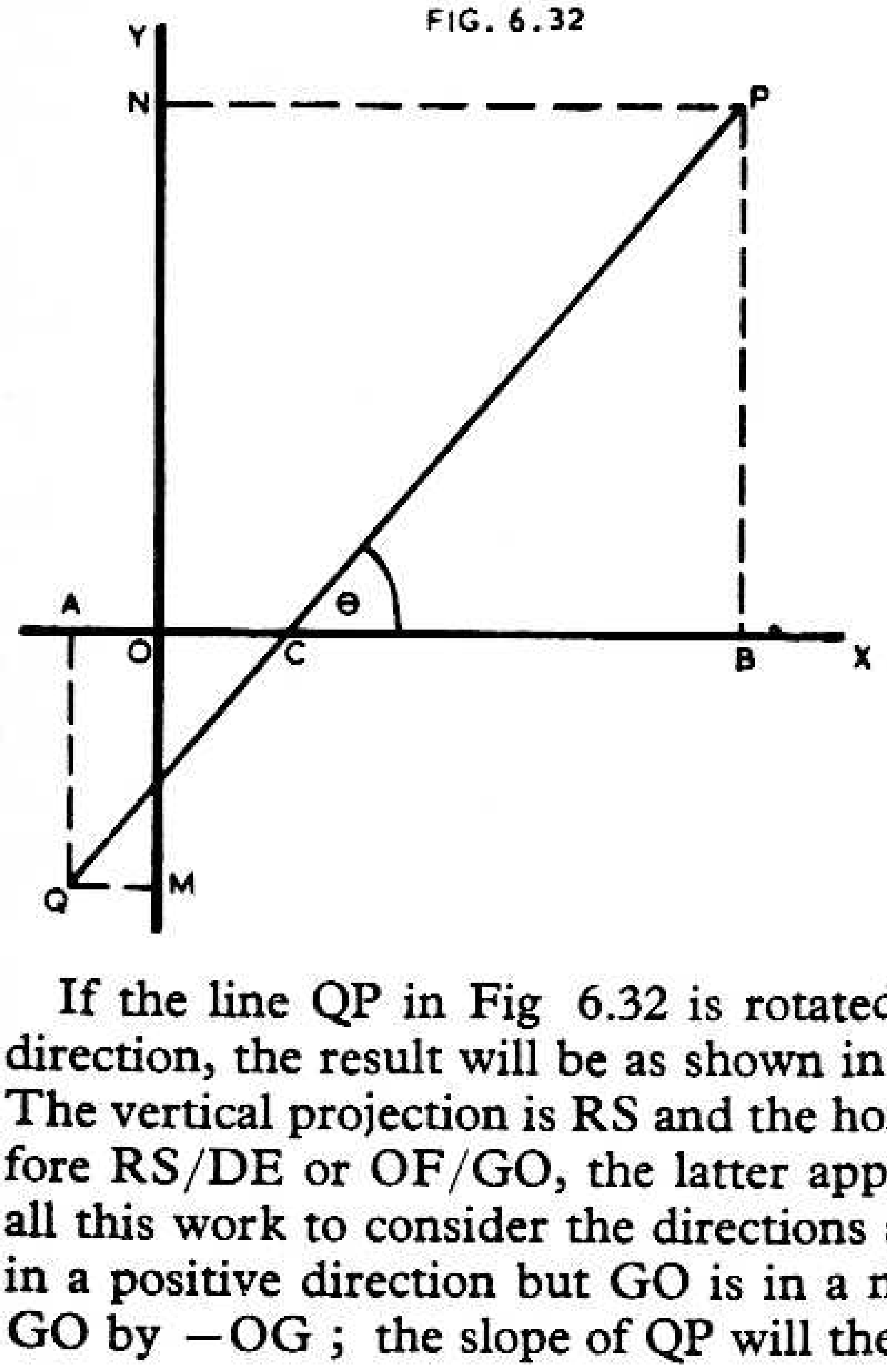
(i) Slope and rate of change

The **slope** of any straight line is the ratio of the lengths of the vertical and horizontal projections of any segment of the line. In Fig. 6.32 the line QP has its vertical projection MN and horizontal projection AB; its slope is therefore MN/AB. Its slope could equally well be based on the projections of its segment CP, and its slope BP/CB. In both cases the result is the same, and is equal to the tangent of the angle of inclination which the line makes with the horizontal axis—

$$\text{slope} = MN/AB = BP/CB = \tan\theta.$$

The lengths of the projections must be measured in terms of the scales to which the line is drawn. Thus horizontal projections such as CB must be measured, not in inches, but in the equivalent number of units corresponding to the length CB on the X axis. Similarly with the vertical projections on the Y axis, which usually has quite a different scale from that for the X axis.

FIG. 6.32



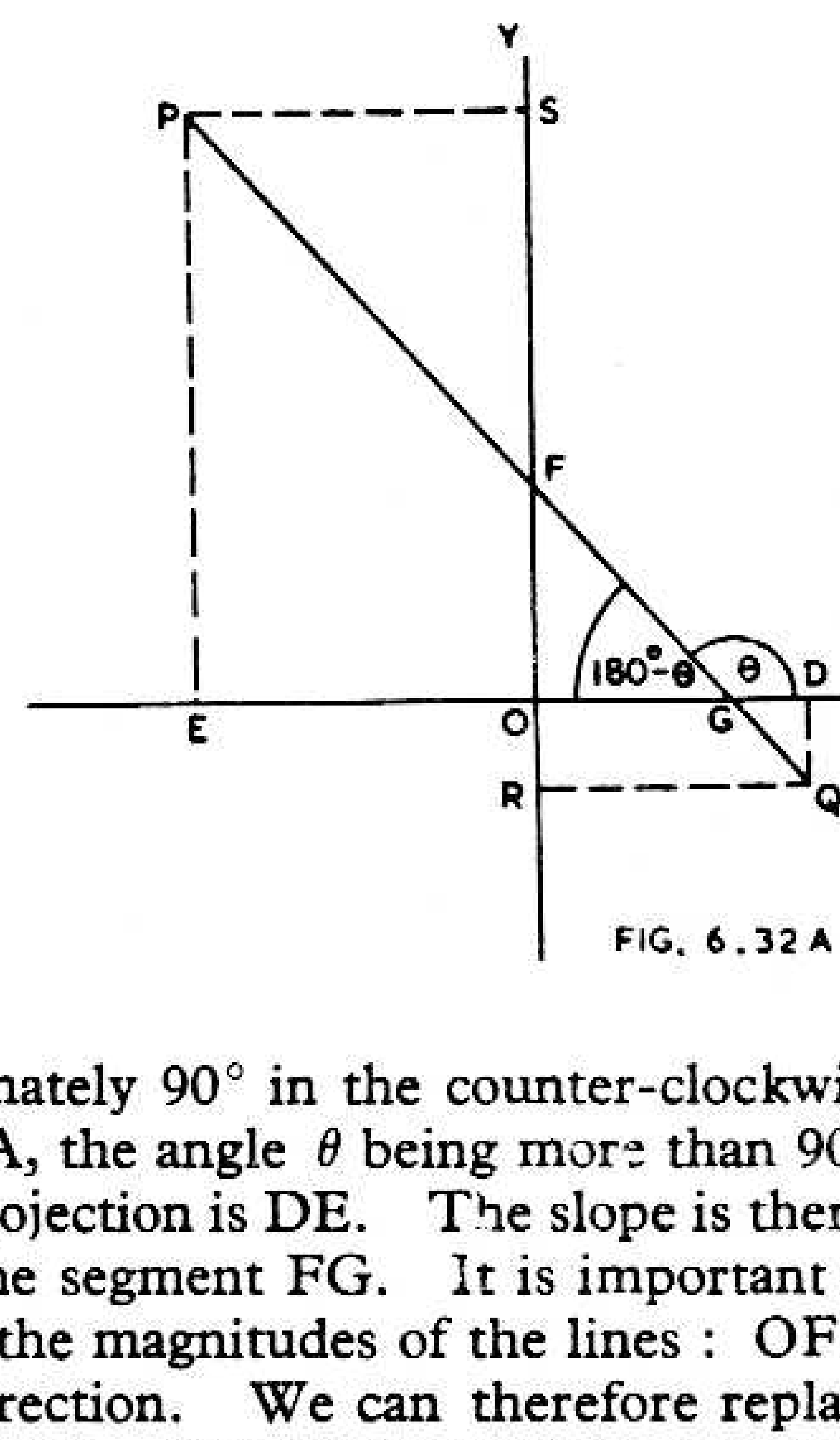
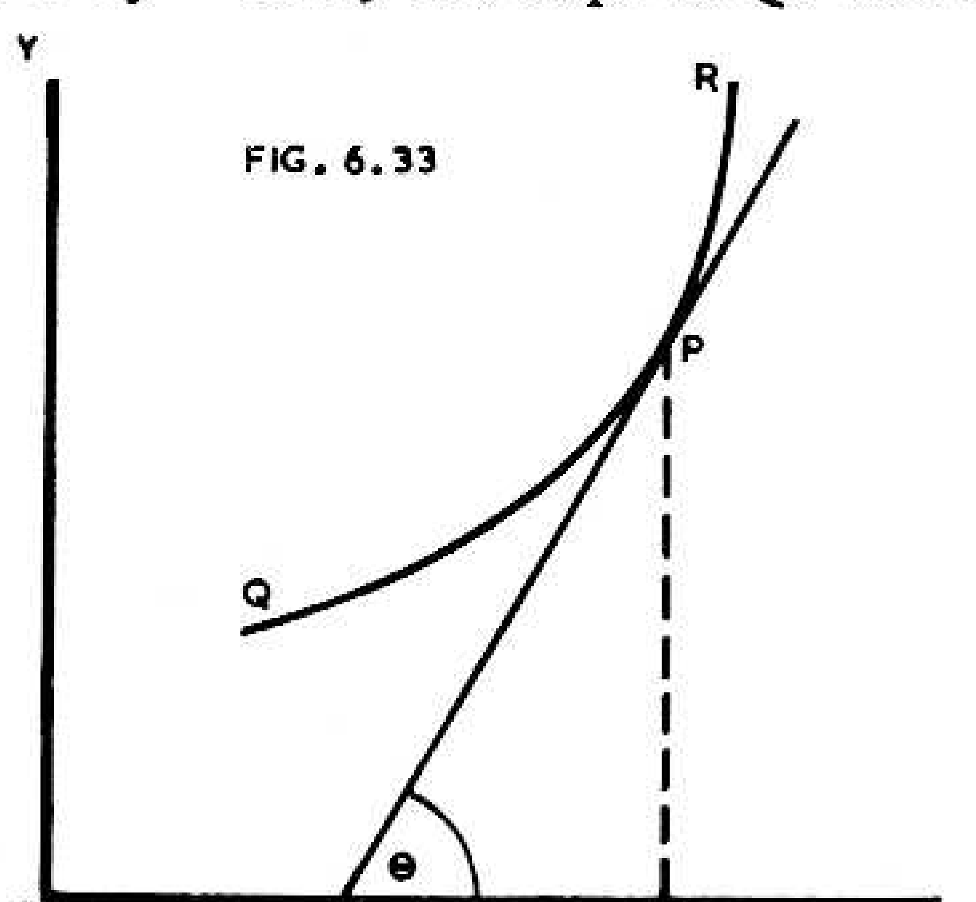


FIG. 6.32 A

If the line QP in Fig 6.32 is rotated approximately 90° in the counter-clockwise direction, the result will be as shown in Fig. 6.32A, the angle θ being more than 90°. The vertical projection is RS and the horizontal projection is DE. The slope is therefore RS/DE or OF/GO, the latter applying to the segment FG. It is important in all this work to consider the directions as well as the magnitudes of the lines: OF is in a positive direction but GO is in a negative direction. We can therefore replace GO by $-$OG; the slope of QP will then be given by $-$(OF/OG) which is described as a negative slope. The loadlines on valve plate characteristics are examples of lines with negative slope (e.g. Fig. 2.22).



When a line is curved, its slope varies from point to point. The slope at any point is given by the slope of the tangent at that point. In Fig. 6.33 the curve is QPR, and the slope at P is given by BP/CB or $\tan\ \theta$.

Rate of change

One of the most important relationships between the variables in any law, formula or equation is the rate of change of the whole function with its independent variable.

Definition

The rate of change is the amount of change in the function, per unit change in the value of the independent variable. The rate of change of the function is therefore the ratio of change in the function to the change in the variable which produces it.

Consider the equation for a straight line

$$y = ax + b$$

where a and b are constants. Let us take two values of x, one equal to x_1 and the other $(x_1 + \Delta x)$, where Δx is a small increment of x.

Point 1 : $x = x_1$ $\therefore y_1 = ax_1 + b$ (1)

Point 2 : $x = (x_1 + \Delta x)$ $\therefore y_2 = a(x_1 + \Delta x) + b$ (2)

Subtracting (1) from (2), $(y_2 - y_1) = a.\Delta x$

Putting $(y_2 - y_1) = \Delta y$, $\Delta y = a.\Delta x$

Dividing both sides by Δx, $\dfrac{\Delta y}{\Delta x} = a$ (3)

Here Δy is the amount of change in the function for a change Δx in the independent variable. The " rate of change " is defined as the amount of change in the function per unit change in the independent variable, that is

$$\text{rate of change} = \Delta y/\Delta x \quad (4)$$

Referring to Fig. 6.34, we have a graph of $y = ax + b$ which is, of course, a straight line cutting the Y axis at a height b above the origin. The first point (x_1, y_1) is at P, and the second point $(x_1 + \Delta x, y_1 + \Delta y)$ is at Q. In the preceding argument we found that $\Delta y/\Delta x = a$. In Fig. 6.34 $\Delta x =$ PR and $\Delta y =$ RQ, so that $\Delta y/\Delta x =$ RQ/PR, which is the slope of the line $y = ax + b$. Thus

$$\text{slope} = \frac{\text{RQ}}{\text{PR}} = \frac{\Delta y}{\Delta x} = a \quad (5)$$

Equations (4) and (5) prove that **the rate of change is the same as the slope for a straight line.**

We now proceed to consider the general case when the function is not linear. In Fig. 6.35 there is plotted a function which may be of the form

$$y = ax^2 + bx + c$$

where a, b and c are constants. As before, the independent variable (x) is plotted horizontally, while the dependent variable (y) is plotted vertically. The vertical coordinate of any point P (i.e. AP in Fig. 6.35) represents, to its proper scale, the value of $ax^2 + bx + c$ which is a function of x, while the horizontal coordinate (OA) represents, to its own scale, the value of x.

FIG. 6.34

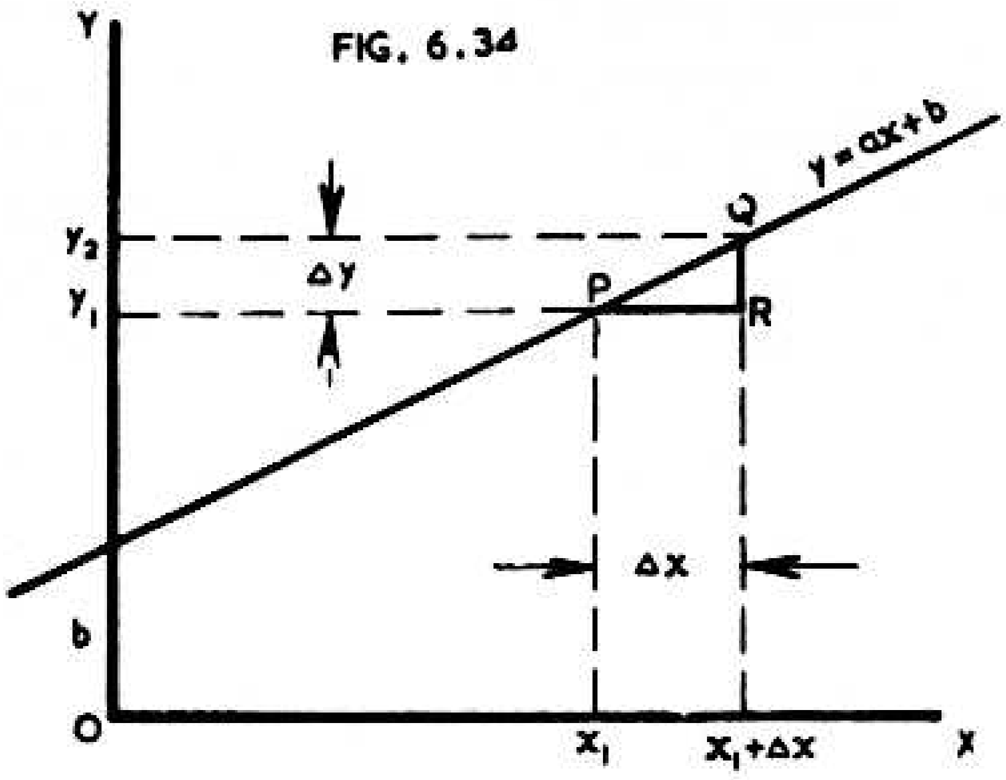
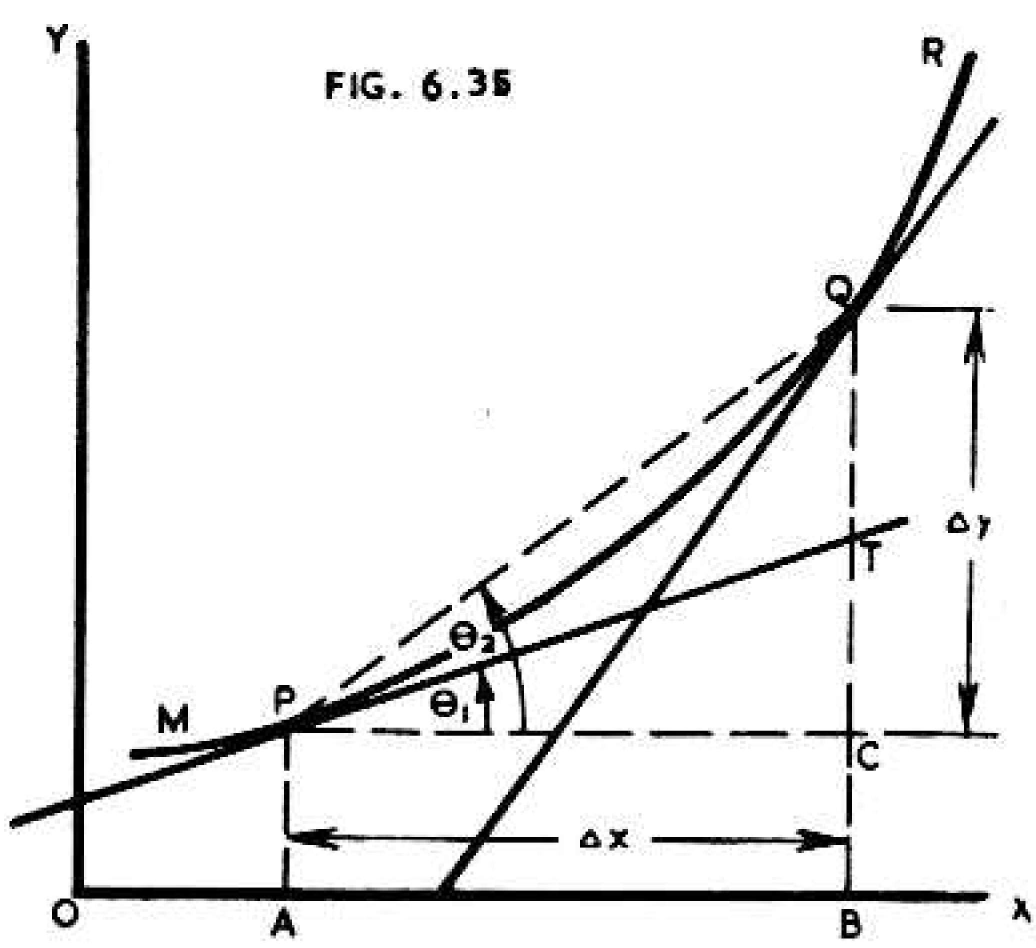


FIG. 6.35



As with the simpler case of the straight line, take a second point (Q) with its x coordinate increased by Δx. Also, as before, call the increment in the y coordinate Δy. It will therefore be seen that, commencing at point P, an increment ($\Delta x =$ PC) in the value of x results in an increment ($\Delta y =$ CQ) in the value of y.

The **average rate of change** over this increment is, as before, $\Delta y/\Delta x$ which is the slope of the chord PQ and the tangent of the angle θ_2 which PQ makes with the horizontal. If now, leaving point P unchanged, we gradually move Q along the curve towards P we will see that, as Q approaches P, θ_2 approaches θ_1 until in the limiting case the slope of the chord approaches the slope of the tangent (PT). The tangent to a curve at any point shows the instantaneous slope of the curve at that point and therefore the instantaneous rate of increase of the function at that point.

(ii) Differentiation

We may express the foregoing argument in the mathematical form of limits—

$$\underset{\Delta x \to 0}{Lt} \quad (\Delta y/\Delta x) = \tan\,\theta_1 \qquad (6)$$

which says that the limit (as Δx is made smaller and approaches zero) of $\Delta y/\Delta x$ is tan θ_1 or the slope of the tangent PT, which is the instantaneous rate of increase at point P.

This is given the symbol dy/dx which is " the **differential coefficient** (or derivative) of y with regard to x," and is spoken of as " dee y by dee x."

It should be noted that dy/dx is a single symbol, not a fraction, and is merely a short way of writing

$$\underset{\Delta x \to 0}{Lt} \quad (\Delta y/\Delta x).$$

Differentiation is the process of finding the differential coefficient (or derivative). Some examples are given below and in each case the result may be obtained by considering the increase in the function which results from an increase Δx in the independent variable.

Note : u and v are functions of x ; a, b and c are constants.

1. Derivative of a constant $[y = c]$ $\quad dy/dx = 0$ (7)
2. Derivative of a variable with respect to itself
   $[y = x]$: $\quad dy/dx = dx/dx = 1$ (8)
3. Derivative of a variable multiplied by a constant
   $[y = cx]$: $\quad dy/dx = c$ (9)
4. Derivative of powers of a variable
   $[y = x^2]$: $\quad dy/dx = 2x$ (10)
   $[y = x^3]$: $\quad dy/dx = 3x^2$ (11)
   $[y = x^4]$: $\quad dy/dx = 4x^3$ (12)
   $[y = x^n]$: $\quad dy/dx = nx^{n-1}$ (13)
   This applies for n negative as well as positive.
5. Derivative of a constant times a function of a variable
   $[y = cx^2]$: $\quad dy/dx = 2cx$ (14)
   $[y = c.u]$: $\quad dy/dx = c.du/dx$ (15)
6. Derivative of fractional powers of a variable
   $[y = x^{\frac{1}{2}}]$: $\quad dy/dx = \frac{1}{2}x^{-\frac{1}{2}}$ (16)
   $[y = x^{1/n}]$: $\quad dy/dx = (1/n)x^{(1/n)-1}$ (17)
7. Derivative of a sum or difference
   $[y = u \pm v]$: $\quad dy/dx = du/dx \pm dv/dx$ (18)
   $[y = ax^3 + bx^2 - cx]$: $\quad dy/dx = 3ax^2 + 2bx - c$ (19)
8. Derivative of a product of two functions
   $[y = u.v]$: $\quad dy/dx = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ (20)
   $[y = (x + 1)\,x^2]$: $\quad dy/dx = (x + 1).2x + x^2.1 = 3x^2 + 2x$ (21)
9. Derivative of a quotient of two functions
   $[y = u/v]$: $\quad \dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$ (22)
10. Differentiation of a function of a function
   $[y = F(u)$ where $u = F(x)]$: $\quad \dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$ (23)
   $[y = u^2$ where $u = ax^2 + b]$: $\quad dy/dx = 2(ax^2 + b)(2ax)$ (24)

11.	$[y = \epsilon^x]$:	$dy/dx = \epsilon^x$	(25)
	$[y = a.\epsilon^{mx}]$:	$dy/dx = a.m.\epsilon^{mx}$	(26)
	$[y = \epsilon^u]$:	$dy/dx = \epsilon^u . du/dx$	(27)
12.	$[y = \log_\epsilon x]$:	$dy/dx = 1/x$	(28)
	$[y = \log_\epsilon u]$:	$dy/dx = (1/u)(du/dx)$	(29)
	$[y = \log_{10} u]$:	$dy/dx = (1/u)(du/dx) \log_{10} \epsilon$	(30)
		$= (0.4343)(1/u)(du/dx)$	(31)
13.	$[y = \sin x]$:	$dy/dx = \cos x$	(32)
	$[y = \cos x]$:	$dy/dx = - \sin x$	(33)
	$[y = \tan x]$:	$dy/dx = \sec^2 x$	(34)
	$[y = \cot x]$:	$dy/dx = - \operatorname{cosec}^2 x$	(35)
	$[y = \sec x]$:	$dy/dx = \sec x . \tan x$	(36)
	$[y = \operatorname{cosec} x]$:	$dy/dx = - \operatorname{cosec} x . \cot x$	(37)
14.	$[y = \sin^{-1} x]$:	$dy/dx = 1/\sqrt{1 - x^2}$*‡	(38)
	$[y = \cos^{-1} x]$:	$dy/dx = - 1/\sqrt{1 - x^2}$†‡	*(39)
	$[y = \tan^{-1} x]$:	$dy/dx = 1/(1 + x^2)$‡	(40)
15.	$[y = \sinh x]$:	$dy/dx = \cosh x$	(41)
	$[y = \cosh x]$:	$dy/dx = \sinh x$	(42)
	$[y = \tanh x]$:	$dy/dx = \operatorname{sech}^2 x.$	(43)

Successive differentiation

If y is a function of x, then dy/dx will also be a function of x and can therefore be differentiated with respect to x, giving

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) \text{ which is written as } \frac{d^2y}{dx^2}$$

and spoken of as " dee two y by dee x squared." Here again this is only a symbol which must be handled in accordance with its true meaning. The same procedure may be applied again and again.

Example : $y = x^n$ . . . . function $F(x)$

$\frac{dy}{dx} = nx^{n-1}$ . . . . derivative $F'(x)$

$\frac{d^2y}{dx^2} = n(n-1)x^{n-2}$ . . . . . . second derivative $F''(x)$

$\frac{d^3y}{dx^3} = n(n-1)(n-2)x^{n-3}$ . . . . third derivative $F'''(x)$

Application of differentiation

The plate current versus grid voltage characteristic of a triode valve (for constant plate voltage) is a function of the grid voltage, and follows approximately the law

$$I_p = K(\mu E_g + E_p)^{3/2}$$

where K, μ and E_p are constants. The derivative with regard to E_g is dI_p/dE_g, which is the mutual conductance. The second derivative is the rate of change of the mutual conductance with regard to E_g, and is useful when we want to find the conditions for maximum or minimum mutual conductance.

In Fig. 6.36 there is a curve with a **maximum** at point M and a **minimum** at point N. It will be seen that the instantaneous slope of the curve at both points M and N is zero, that is $dy/dx = 0$.

Part of curve :	P to M	M	M to N	N	N to Q
Slope (dy/dx) :	+ ve	O	−ve	O	+ ve
d^2y/dx^2 :		− ve		+ ve	

A maximum is indicated by : $\begin{cases} dy/dx = 0 \\ d^2y/dx^2 \text{ negative} \end{cases}$

A minimum is indicated by : $\begin{cases} dy/dx = 0 \\ d^2y/dx^2 \text{ positive} \end{cases}$

‡ x measured in radians.

*Positive sign if $\sin^{-1} x$ lies in first or fourth quadrant, negative sign if $\sin^{-1} x$ lies in second or third quadrant.

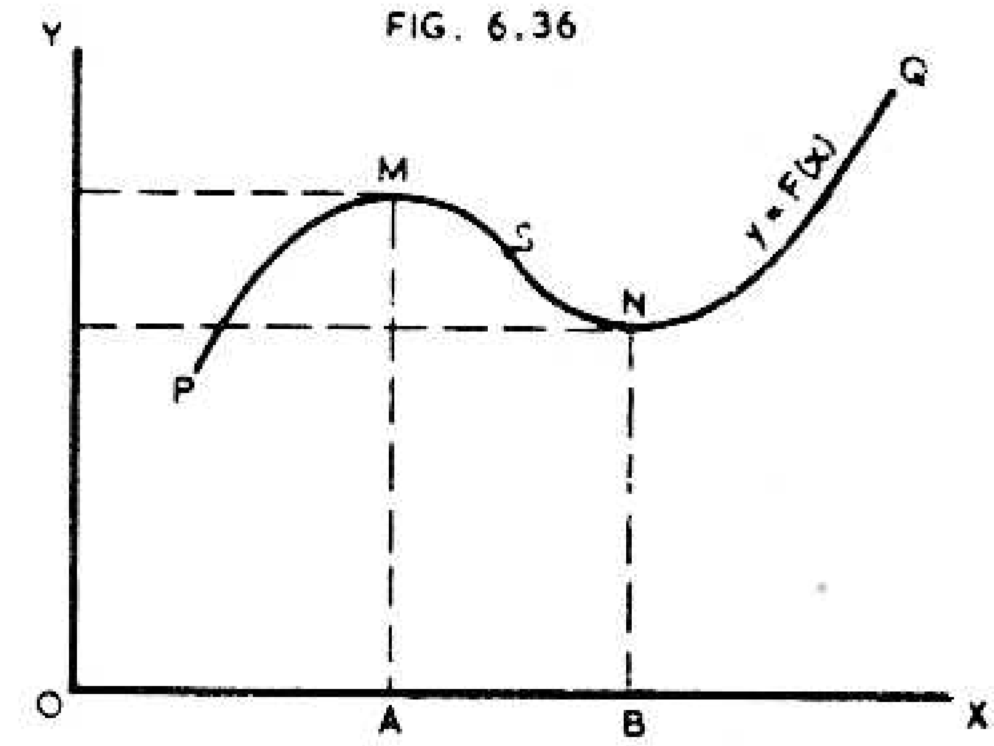
†Negative sign if $\cos^{-1} x$ lies in first or second quadrant, positive sign if $\cos^{-1} x$ lies in third or fourth quadrant.

A point of inflection* is indicated by : $d^2y/dx^2 = 0$

Curve concave upwards indicated by : d^2y/dx^2 positive

Curve concave downwards indicated by : d^2y/dx^2 negative

FIG. 6.36



Examples

(1) To find the maximum value of $y = 2x - x^2 + 4$

$$dy/dx = 2 - 2x$$

For a maximum $dy/dx = 0$, therefore $2 - 2x = 0$, and $x = 1$.

This is the value of x at which a maximum, or a minimum, or a point of inflection occurs.

To see which it is, take the second derivative—

$$d^2y/dx^2 = -2 \text{ which is negative.}$$

Therefore the point is a maximum.

To find the value of y at this point, put the value $(x = 1)$ into $y = 2x - x^2 + 4$.

Therefore $y = 2 - 1 + 4 = 5$.

(2) To find the points of inflection in the curve

$$y = x^4 - 6x^2 - x + 16.$$
$$dy/dx = 4x^3 - 12x - 1$$
$$d^2y/dx^2 = 12x^2 - 12.$$

For points of inflection, $d^2y/dx^2 = 0$, therefore $12x^2 - 12 = 0$

Therefore $12x^2 = 12$, Thus $x = \pm 1$.

There are thus two points of inflection, one at $x = +1$, the other at -1. The values of y at these points are given by substituting these values of x in the function :

$$x = +1 : y = 1 - 6 - 1 + 16 = +10.$$
$$x = -1 : y = 1 - 6 + 1 + 16 = +12.$$

It is always wise to make a rough plot of the curve to see its general shape. Some curves have more than one value of maximum and minimum.

Partial differentiation

Partial differential coefficients, designated in the form $\partial y/\partial x$ (the symbol ∂ may be pronounced " der " to distinguish from " d " in dy/dx) are used in considering the relationship between two of the variables in systems of more than two variables such as the volume of an enclosure having rectangular faces, the sides being of length x, y and z respectively :

$$v = x\,y\,z \quad (44)$$

Thus, the rate of change of volume with the change in length of the side x, while the sides y and z remain constant, is

$$\partial v/\partial x = y\,z \quad (45)$$

Similarly $\partial v/\partial y = z\,x$, where z and x are constant (46)

And $\partial v/\partial z = xy$ where x and y are constant (47)

In three-dimensional differential geometry, the equation representing a surface may be represented generally in the form

$$y = F(x, z) \quad (48)$$

In this case, the partial differential coefficient $\partial y/\partial x$ represents the slope at the point (x, y, z) of the tangent to the curve of intersection of the surface with a plane parallel to the plane passing through the x and y axes and separated by a fixed distance z from the latter.

Thus $\partial y/\partial x$ represents the slope of a tangent to a cross section of a three-dimensional solid, the partial derivative reducing the three-dimensional body to a form suitable for two-dimensional consideration. " $\partial y/\partial x$ " is equivalent to " dy/dx (z constant) " when there are three variables, x, y and z.

*A point of inflection is one at which the curvature changes from one direction to the other (e.g. S in Fig. 6.36). It is necessarily a point of maximum or minimum slope.

Partial differentials are therefore particularly valuable in representing Valve Coefficients [see Chapter 2 Sect. 9(ix)].

Total differentiation

When there are three independent variables (x, y, z) which are varying simultaneously but independently of each other,

$$u = F(x,y,z) \tag{49}$$

the total differential is

$$du = \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy + \frac{\partial u}{\partial z}\, dz \tag{50}$$

and similarly for two, or any larger number of independent variables.

When the independent variables are functions of a single independent variable (t) the total differential with respect to t is

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt} \tag{51}$$

(iii) Integration

Integration is merely the inverse of differentiation. For example—

Differentiation— $\frac{d}{dx}(4x^3 + 2) = 12x^2$

Integration— $\int 12x^2\, dx = 4x^3 + C$

The sign $\int$ (called " integral ") before a quantity indicates that the operation of integration is to be performed on the expression which follows.

The dx which follows the expression is merely a short way of writing " with respect to x." Just as the constant 2, in the function above, disappeared during the process of differentiation, so it is necessary to replace it in the inverse procedure of integration. But when we are given the integral alone, we do not know what was the value of the constant, so we add an unknown constant C, the value of which may be determined in some cases from other information available.

Useful rules for integrals

a, b, c = constants ; C = constant of integration ; u and v are functions of x.

1. $\int a.F(x)\, dx = a\int F(x)\, dx$; $\int a\, dx = ax + C$ (52)

2. $\int (u \pm v)\, dx = \int u\, dx \pm \int v\, dx$ (similarly for more than two) (53)

3. $\int x^n\, dx = \frac{1}{n+1} x^{n+1} + C \qquad (n \neq -1)$ (54)

4. $\int a^x\, dx = (a^x/\log_\epsilon a) + C$ (55)

5. $\int \epsilon^x\, dx = \epsilon^x + C$; $\int \epsilon^{ax}\, dx = (1/a)\,\epsilon^{ax} + C$ (56)

   $\int x\epsilon^x dx = \epsilon^x(x - 1) + C$; $\int x^m \epsilon^x dx = x^m \epsilon^x - m\int x^{m-1} e^x dx + C$* (57)

6. $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$ (integration by parts) (58)

7. $\int \frac{dx}{x} = \int \frac{1}{x}\, dx = \log_\epsilon x + C = \log_\epsilon c\, x$ (59)

8. $\int a^x \log_\epsilon a\, dx = a^x + C$; $\int \log_a x\, dx = x \log_a (x/\epsilon) + C$ (60)

9. $\int (ax + b)^n\, dx = \frac{(ax + b)^{n+1}}{a\,(n + 1)} + C \qquad (n \neq -1)$ (61)

* $m > 0$.

10. $\int \frac{dx}{ax + b} = \int \frac{1}{ax + b} dx = (1/a) \log_\epsilon (ax + b) + C$ (62)

11. $\int \frac{x\, dx}{ax + b} = \int \frac{x}{ax + b}\, dx = (1/a^2)\, [ax + b - b \log_\epsilon (ax + b)] + C$ (63)

12. $\int \frac{x\, dx}{(ax + b)^2} = \int \frac{x}{(ax + b)^2}\, dx = \frac{1}{a^2}\left[\frac{b}{ax + b} + \log_\epsilon (ax + b)\right] + C$ (64)

13. $\int \frac{x^2\, dx}{ax + b} = \int \frac{x^2}{ax + b}\, dx = \frac{1}{a^3}\left[\frac{(ax + b)^2}{2} - 2b(ax + b) + b^2 \log_\epsilon (ax + b)\right] + C$ (65)

14. $\int \frac{dx}{x^2 + a^2} = \int \frac{1}{x^2 + a^2}\, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C = -\frac{1}{a} \cot^{-1} \frac{x}{a} + C$ (66)

15. $\int \frac{dx}{x^2 - a^2} = \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \frac{x - a}{x + a} + C = \frac{1}{2a} \log_\epsilon \frac{a - x}{a + x} + C$ (67)

16. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1} (x/a) + C = -\cos^{-1} (x/a) + C$ (68)

17. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \int \frac{1}{\sqrt{x^2 \pm a^2}}\, dx = \log_\epsilon (x + \sqrt{x^2 \pm a^2}) + C$ (69)

18. $\int \sin ax\, dx = -(1/a) \cos ax + C$ (70)

19. $\int \cos ax\, dx = (1/a) \sin ax + C$ (71)

20. $\int \tan ax\, dx = -(1/a) \log_\epsilon \cos ax + C = (1/a) \log_\epsilon \sec ax + C$ (72)

21. $\int \operatorname{cosec} ax\, dx = (1/a) \log_\epsilon (\operatorname{cosec} ax - \cot ax) + C$ (73)

$= (1/a) \log_\epsilon \tan (ax/2) + C$ (74)

22. $\int \sec ax\, dx = (1/a) \log_\epsilon (\sec ax + \tan ax) + C$ (75)

$= (1/a) \log_\epsilon \tan [(ax/2) + \pi/4] + C$ (76)

23. $\int \cot ax\, dx = (1/a) \log_\epsilon \sin ax + C = -(1/a) \log_\epsilon \operatorname{cosec} ax + C$ (77)

24. $\int \sin^2 ax\, dx = x/2 - (1/2a) \sin ax \cos ax + C$ (78)

$= x/2 - (1/4a) \sin 2ax + C$ (79)

25. $\int \cos^2 ax\, dx = x/2 + (1/4a) \sin 2ax + C$ (80)

26. $\int \sinh x\, dx = \cosh x$ (81)

27. $\int \cosh x\, dx = \sinh x$ (82)

Rules to assist integration

1. If the function is the sum of several terms, or can be put into this form, integrate term by term.
2. If the function is in the form of a product or a power, it is usually helpful to multiply out or expand before integrating.
3. Fractions may be either divided out, or written as negative powers.
4. Roots should be treated as fractional powers.
5. A function of x may be replaced by u, and then

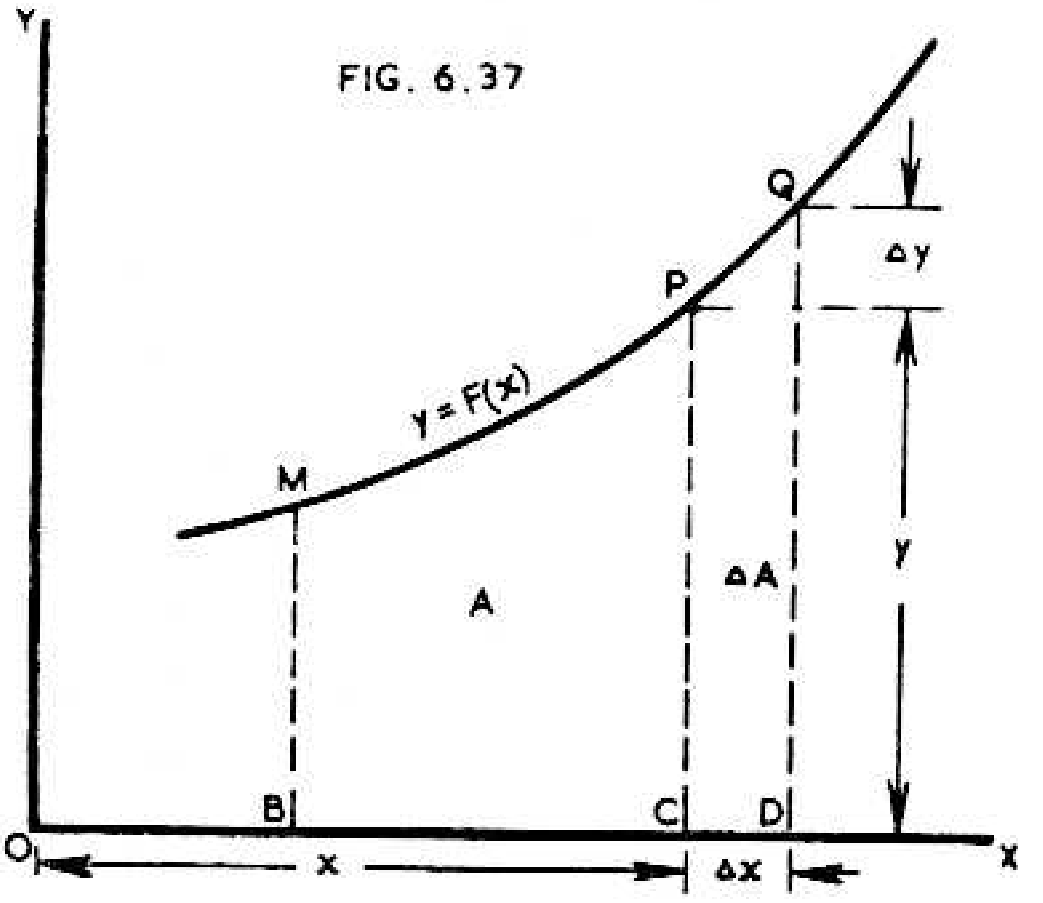
$$\int F(x)\, dx = \int F(x) \frac{dx}{du}\, du.$$

Areas by integration

If the area under a given portion of a curve is A (Fig. 6.37), then a small increase Δx on the horizontal axis causes an increase ΔA in area, where

$$\Delta A = \Delta x(y + \tfrac{1}{2}\Delta y).$$

As Δx and Δy are made smaller, in the limiting case as Δx approaches zero, the value of $(y + \frac{1}{2}\Delta y)$ approaches y,



i.e.

$$\underset{\Delta x \to 0}{Lt}\left[\frac{\Delta A}{\Delta x}\right] = y$$

$$\text{Therefore } \frac{dA}{dx} = y = F(x)$$

$$\text{Therefore } dA = y.dx = F(x)\,dx$$

$$\text{Thus } A = \int y\,dx = \int F(x)\,dx \qquad (83)$$

The area is therefore given by the integral of the function, over any desired range of values of x.

Example

To find the area under the curve $y = 3x^2$ from $x = 1$ to $x = 4$.

$$A = \int 3x^2\,dx = x^3 + C$$

when $x = 1$, $A = 0$, therefore $x^3 + C = 0$, therefore $C = -1$
when $x = 4$, $A = x^3 - 1 = 4^3 - 1 = 63$.

Definite integrals

When it is desired to indicate the limits in the value of x between which the integral is desired, the integral is written, as for the example above,

$$\int_{x=1}^{x=4} 3x^2\,dx \text{ or } \int_1^4 3x^2\,dx.$$

These limits are called the **limits of integration**, and the integral is called the **definite integral**. For distinction, the unlimited integral is called the **indefinite integral.**

The definite integral is the difference between the values of the integral for $x = b$ and $x = a$ —

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(x = b) - F(x = a) \qquad (84)$$

Owing to the subtraction, the constant of integration does not appear in definite integrals.

Special properties of definite integrals

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \qquad (85)$$

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx \qquad (86)$$

Examples :

$$\int_0^{\pi/2} \sin\theta\,d\theta = \Big[-\cos\theta\Big]_0^{\pi/2} = [-\cos\pi/2 + \cos 0] = 0 + 1 = 1 \qquad (87)$$

$$\int_0^{\pi/2} \cos\theta\, d\theta = \Big[\sin\theta\Big]_0^{\pi/2} = [\sin\pi/2 - \sin 0] = 1 - 0 = 1 \qquad (88)$$

$$\int_0^{\pi} \sin\theta\, d\theta = \Big[-\cos\theta\Big]_0^{\pi} = [-\cos\pi + \cos 0] = 1 + 1 = 2 \qquad (89)$$

$$\int_0^{\pi} \cos\theta\, d\theta = \Big[\sin\theta\Big]_0^{\pi} = [\sin\pi - \sin 0] = 0 - 0 = 0 \qquad (90)$$

$$\int_0^{2\pi} \sin\theta\, d\theta = \Big[-\cos\theta\Big]_0^{2\pi} = [-\cos 2\pi + \cos 0] = -1 + 1 = 0 \qquad (91)$$

$$\int_0^{2\pi} \cos\theta\, d\theta = \Big[\sin\theta\Big]_0^{2\pi} = [\sin 2\pi - \sin 0] = 0 - 0 = 0 \qquad (92)$$

$$\int_0^{\pi} \sin^2 n\theta\, d\theta = \tfrac{1}{2}\Big[\theta - (1/2n)\sin 2n\theta\Big]_0^{\pi} = \pi/2 \qquad (93)$$

$$\int_0^{\pi} \cos^2 n\theta\, d\theta = \int_0^{\pi} (1 - \sin^2 n\theta\, d\theta)\, d\theta = \Big[\theta\Big]_0^{\pi} - \pi/2 = \pi/2 \qquad (94)$$

The following may also be derived* where $m \neq n$

$$\int_0^{2\pi} \sin n\theta\, d\theta = 0 \qquad \int_0^{2\pi} \cos n\theta\, d\theta = 0 \qquad (95)$$

$$\int_0^{2\pi} \sin^2 n\theta\, d\theta = \pi \qquad \int_0^{2\pi} \cos^2 n\theta\, d\theta = \pi \qquad (96)$$

$$\int_0^{2\pi} \sin m\theta \cos n\theta\, d\theta = 0 \qquad \int_0^{2\pi} \sin m\theta \sin n\theta\, d\theta = 0 \qquad (97)$$

$$\int_0^{2\pi} \cos m\theta \cos n\theta\, d\theta = 0 \qquad \int_0^{2\pi} d\theta = 2\pi \qquad (98)$$

$$\int_0^{2\pi} \sin n\theta \cos n\theta\, d\theta = 0 \qquad (99)$$

Average values by definite integral

The value of the definite integral is the area under the curve between the limits on the horizontal (x) axis. The average value of the height is determined by dividing the area by the length, or in other words the average value of y is determined by dividing the definite integral by the difference in the limiting values of x.

Examples :

1. $y = 3x^2$ from $x = 1$ to $x = 4$.

$$y_{av} = \frac{1}{4-1}\int_1^4 3x^2\, dx = \frac{1}{3}\Big[x^3\Big]_1^4 = \frac{1}{3}\Big[64 - 1\Big] = \frac{63}{3} = 21.$$

2. $e = E_m \sin\theta$ from $\theta = 0$ to $\theta = \pi$.

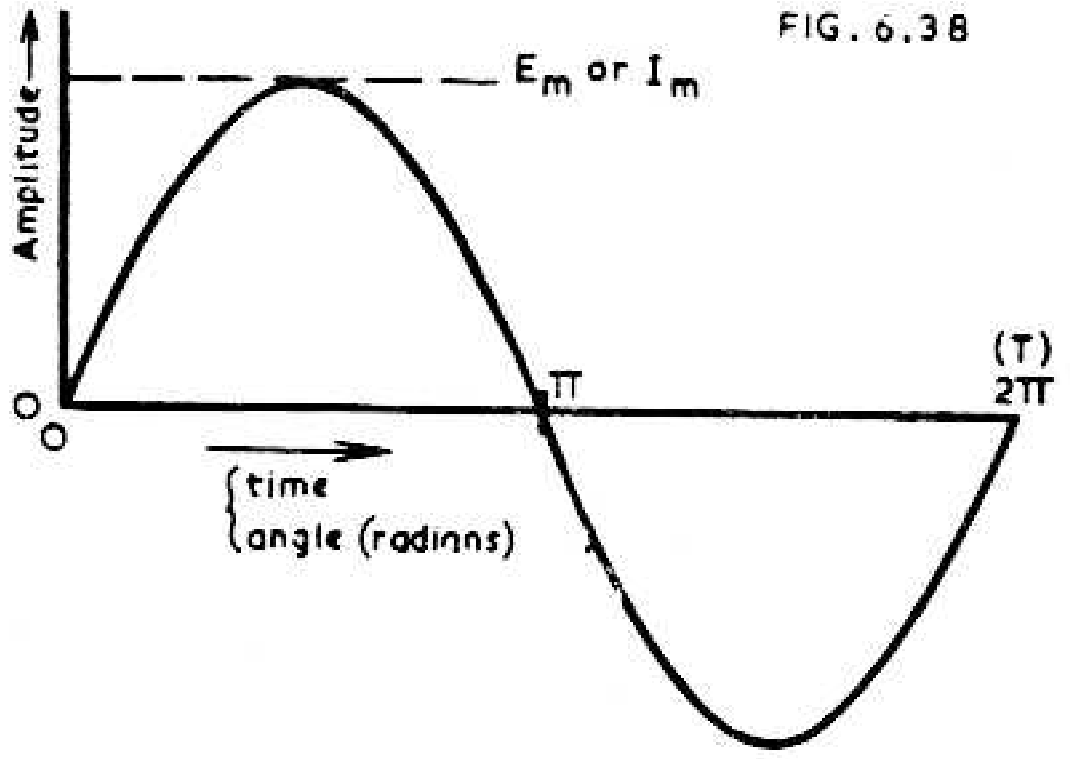
$$E_{av} = E_m \frac{1}{\pi}\int_0^{\pi} \sin\theta\, d\theta = E_m \frac{1}{\pi}\Big[-\cos\theta\Big]_0^{\pi} = \frac{2}{\pi} E_m.$$

*This also applies with limits from k to $(k + 2\pi)$.

This is the average value of a sine wave voltage. It was taken from 0 to π since this is the range over which it is positive. The other half cycle is similar but negative, so that the average over the whole cycle is zero.

3.
$$(I_{rms})^2 = \frac{1}{2\pi}\int_0^{2\pi} (I_m \sin\theta)^2\, d\theta$$
$$= \frac{I_m^2}{2\pi}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{2\pi}$$
$$= \frac{I_m^2}{2\pi}[\pi - 0 - 0 + 0]$$
$$= \frac{I_m^2}{2\pi}\cdot\pi = \frac{I_m^2}{2}.$$

FIG. 6.38



Therefore $I_{rms} = I_m/\sqrt{2} \approx 0.707\, I_m$.

This is the root mean square value of the current with sine waveform.

(iv) Taylor's Series

If $f(x)$ be a function with first derivative $f'(x)$, second derivative $f''(x)$, third derivative $f'''(x)$, etc., and if the function and its first n derivatives are finite and continuous from $x = a$ to $x = b$, then the following expansion holds true in the interval from $x = a$ to $x = b$:

$$f(x) = f(a) + \frac{x-a}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots + \frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{(x-a)^n}{n!}f^n(x_n) \qquad (100)$$

where $a < x_n < b$.

The final term is called the **remainder** ; if this can be made as small as desired by making n sufficiently large, the series becomes a convergent infinite series, converging to the value $f(x)$.

Another form of Taylor's Series is :

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \ldots \ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{n!}f^n(a) + \ldots \qquad (101)$$

The sum of the first few terms of Taylor's Series gives a good approximation to $f(x)$ for values of x near $x = a$.

Examples of the use of Taylor's Series :

(1) To expand $\sin(a + h)$ in powers of h.

$f(a) = \sin a$ and $f(h) = \sin h$

Differentiating in successive steps we get

$f'(a) = \cos a$
$f''(a) = -\sin a$
$f'''(a) = -\cos a$
$f''''(a) = \sin a$

Applying the alternative form of Taylor's Series,

$$\sin(a+h) = \sin a + h\cos a - \frac{h^2}{2!}\sin a - \frac{h^3}{3!}\cos a + \frac{h^4}{4!}\sin a + \ldots \qquad (102)$$

(2) Similarly

$$\cos(a+h) = \cos h - h\sin a - \frac{h^2}{2!}\cos a + \frac{h^3}{3!}\sin a + \frac{h^4}{4!}\cos a + \ldots \qquad (103)$$

(v) Maclaurin's Series

Maclaurin's Series is a special case of Taylor's Series where $a = 0$.

$$f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \ldots \tag{104}$$

The sum of the first few terms of Maclaurin's Series gives a good approximation to $f(x)$ for values of x near $x = 0$.

Example of the use of Maclaurin's Series :

$$f(x) = \cos x$$

then $\cos x = f(0) + \dfrac{f'(0)x}{1!} + \dfrac{f''(0)x^2}{2!} + \dfrac{f'''(0)x^3}{3!} + \dfrac{f''''(0)x^4}{4!}$

where $f(0) = \cos 0 = 1$

$f'(0) = -\sin 0 = 0$

$f''(0) = -\cos 0 = -1$

$f'''(0) = \sin 0 = 0$

$f''''(0) = \cos 0 = 1$

The series may then be written down as

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots \tag{105}$$

Similarly

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots \tag{106}$$

$$j \sin x = j(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots)$$

$$\cos x + j \sin x = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) + j\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)$$

$$= 1 + jx + \frac{j^2x^2}{2!} + \frac{j^3x^3}{3!} + \frac{j^4x^4}{4!} + \ldots \tag{107}$$

Also it may be proved that

$$\log_\epsilon (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \tag{108}$$

and $$\epsilon^{jx} = 1 + jx + \frac{j^2x^2}{2!} + \frac{j^3x^3}{3!} + \frac{j^4x^4}{4!} + \ldots$$

$$= \cos x + j \sin x \text{ (see eqn. 107)} \tag{109}$$

SECTION 8 : FOURIER SERIES AND HARMONICS

(i) Periodic waves and the Fourier Series (ii) Other applications of the Fourier Series (iii) Graphical Harmonic Analysis.

(i) Periodic waves and the Fourier Series

The equation for any periodic wave can be written by substituting the correct values in Fourier's Series :

$$y = F(\theta) = B_o/2 + A_1 \sin \theta + A_2 \sin 2\theta + A_3 \sin 3\theta + \ldots$$
$$+ B_1 \cos \theta + B_2 \cos 2\theta + B_3 \cos 3\theta + \ldots \tag{1}$$

where $B_o/2$ is a constant which is zero if the wave is balanced about the x axis ; its value is the average value of y over one cycle and may be determined by putting $n = 0$ in the expression for B_n below,

$$A_n = \frac{1}{\pi}\int_0^{2\pi} F(\theta) \sin n\theta \, d\theta \text{ where } n = 1,2,3, \text{ etc.}$$

$= 2 \times$ average value of $F(\theta) \sin n\theta$ taken over 1 cycle,

$$B_n = \frac{1}{\pi}\int_0^{2\pi} F(\theta) \cos n\theta \, d\theta \text{ where } n = 0,1,2,3, \text{ etc.}$$

$= 2 \times$ average value of $F(\theta) \cos n\theta$ taken over 1 cycle,

$\theta = \omega t = 2\pi ft$: f = fundamental frequency

$2\theta = 2\omega t = 2\pi(2f)t$: $(2f)$ = second harmonic frequency

and $3\theta = 3\omega t = 2\pi(3f)t$: $(3f)$ = third harmonic frequency

etc.

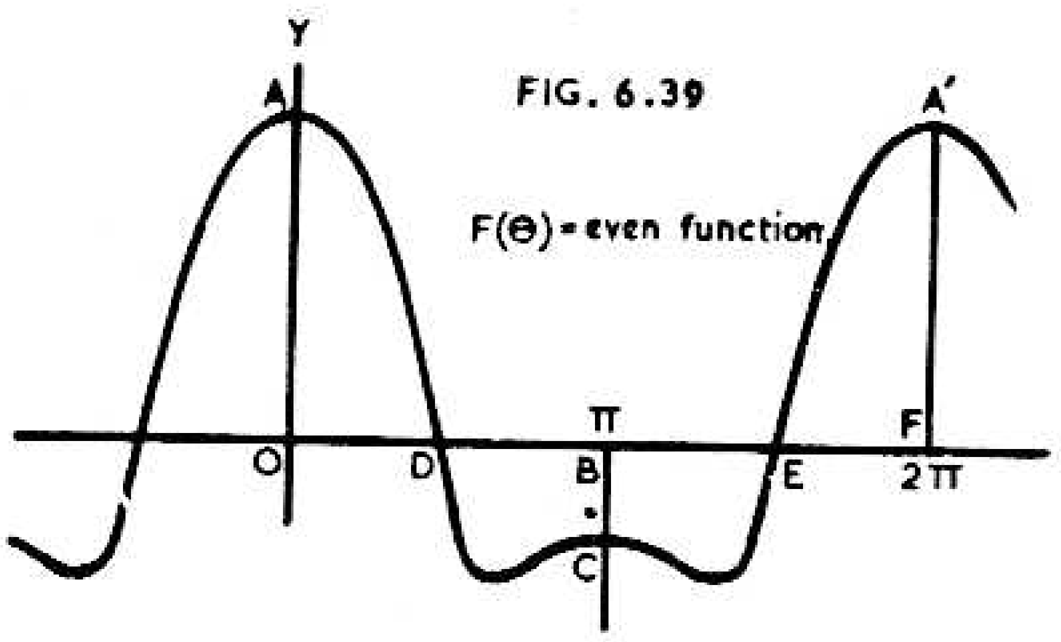


FIG. 6.39

Special cases

(1) $F(\theta)$ is an even function

If the waveform is symmetrical about the y axis (e.g. Fig. 6.39), $F(\theta)$ is called an even function and $A_n = 0$, giving the simplified form

$F(\theta) = B_o/2 + B_1 \cos\theta + B_2 \cos 2\theta + B_3 \cos 3\theta + \ldots$ (2)

This is the equation which applies to all types of distortion introduced by valves.

(2) $F(\theta)$ is an odd function

If the waveform is such that the value of y is equal in magnitude but opposite in sign for plus and minus values of x (e.g. Fig. 6.40), $F(\theta)$ is called an odd function and $B_n = 0$, giving the simplified form :

$F(\theta) = A_1 \sin\theta + A_2 \sin 2\theta + A_3 \sin 3\theta + \ldots$ (3)

FIG. 6.40

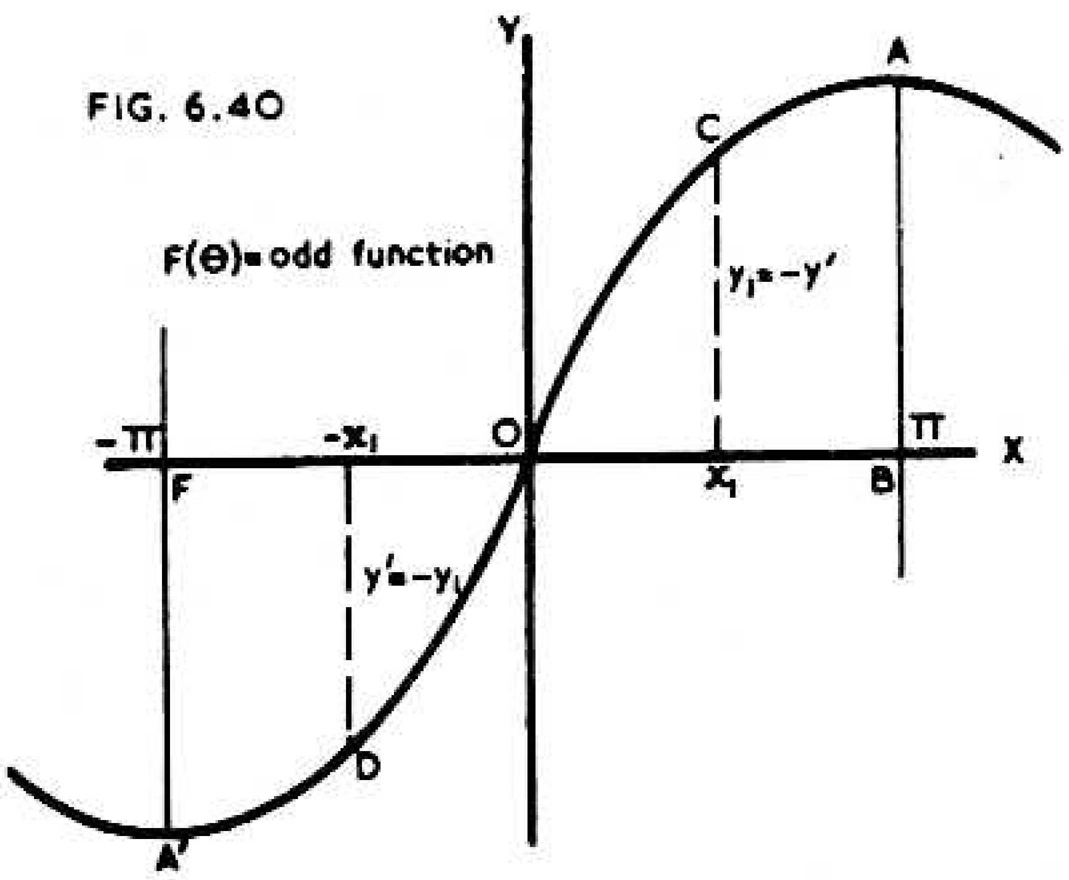
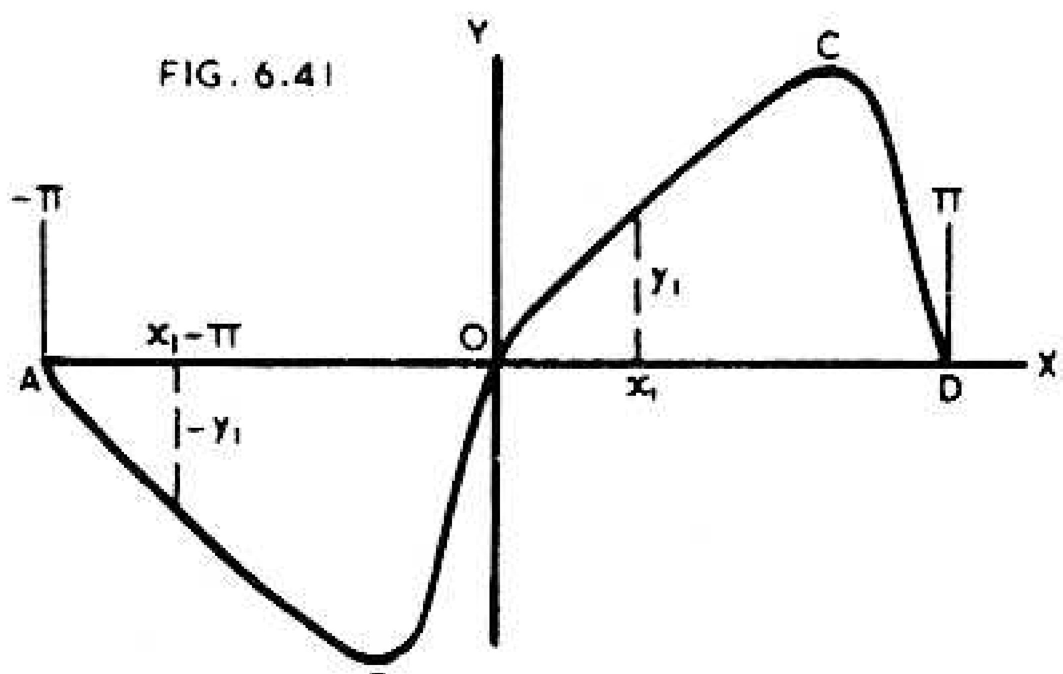


FIG. 6.41



This is the equation for the condition when the fundamental and all the harmonics commence together at zero.

(3) $F(\theta) = -F(\theta \pm \pi)$

If the waveform (Fig. 6.41) is such that the value of y is equal in magnitude but opposite in sign for $x = x_1$ and $x = (x_1 \pm \pi)$, the expansion contains only odd harmonics :

$F(\theta) = A_1 \sin\theta + A_3 \sin 3\theta + A_5 \sin 5\theta + \ldots$

$+ B_1 \cos\theta + B_3 \cos 3\theta + B_5 \cos 5\theta + \ldots$ (4)

(4) $F(\theta)$ is an even function, with the positive and negative portions identical and symmetrical (Fig. 6.42) :

$F(\theta) = B_1 \cos\theta + B_3 \cos 3\theta + B_5 \cos 5\theta + \ldots$ (5)

or if the origin is taken at A,

$F(\theta) = B_1 \sin\theta - B_3 \sin 3\theta + B_5 \sin 5\theta - \ldots$ (6)

This is the equation for a balanced push-pull amplifier.

The general equation (1) **can also be expressed in either of the alternative forms :**

$F(\theta) = B_o/2 + C_1 \sin(\theta + \phi_1) + C_2 \sin(2\theta + \phi_2) + \ldots$ (7)

$$F(\theta) = B_o/2 + C_1 \cos(\theta - \phi_1') + C_2 \cos(2\theta - \phi_2') + \ldots \tag{8}$$

where $C_n = \sqrt{A_n^2 + B_n^2}$
and $\tan \phi_n = B_n/A_n$; $\tan \phi_n' = A_n/B_n$.

The angles $\phi_1, \phi_2, \ldots$ in eqn. (7) are the angles of lead between the harmonics of the sine series and the corresponding sine components in eqn. (1). The angles $\phi_1', \phi_2', \ldots$ in eqn. (8) are the angles of lag between the harmonics of the cosine series and the corresponding cosine components in eqn. (1). All the angles ϕ and ϕ' in equations (7) and (8) are measured on the scales of angles for the harmonics

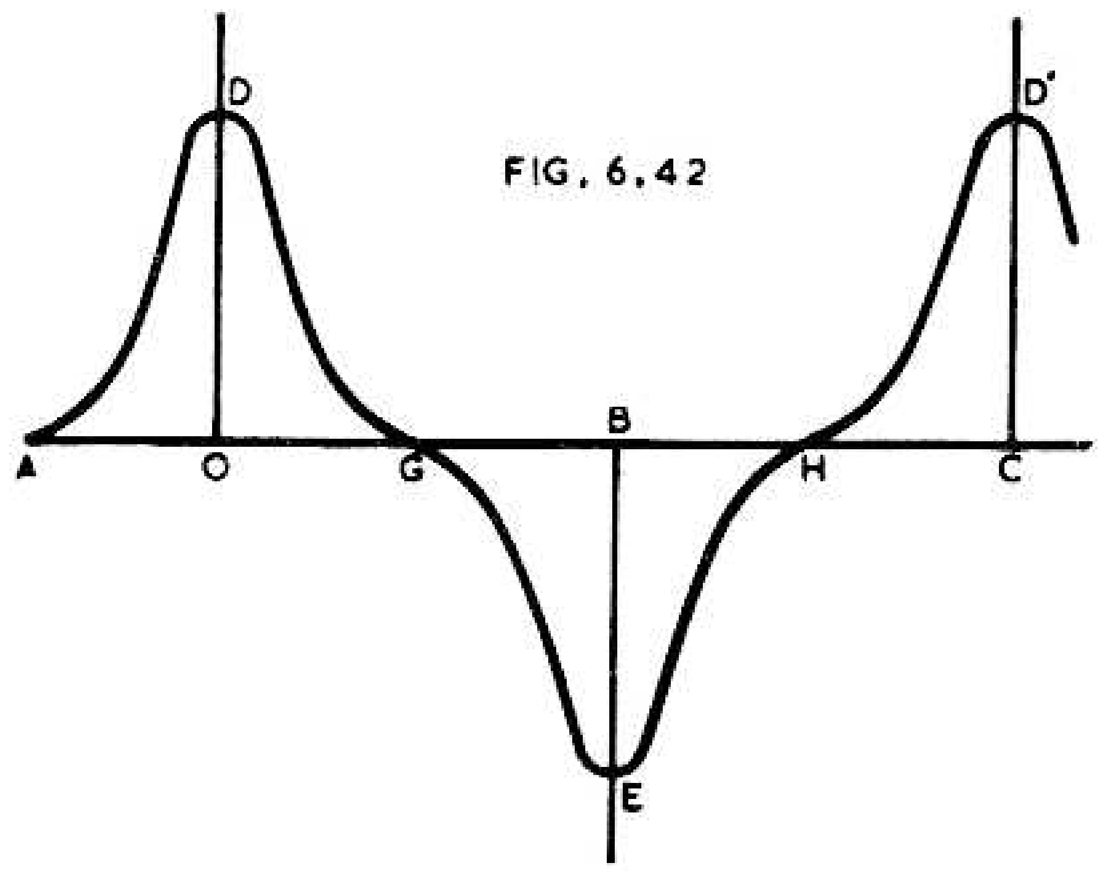


FIG. 6.42

Harmonic composition of some common periodic waves (Fig. 6.43)

Square wave (A)

$$y = \frac{4E}{\pi}\left(\cos \theta - \frac{\cos 3\theta}{3} + \frac{\cos 5\theta}{5} - \frac{\cos 7\theta}{7} + \ldots\right) \tag{9}$$

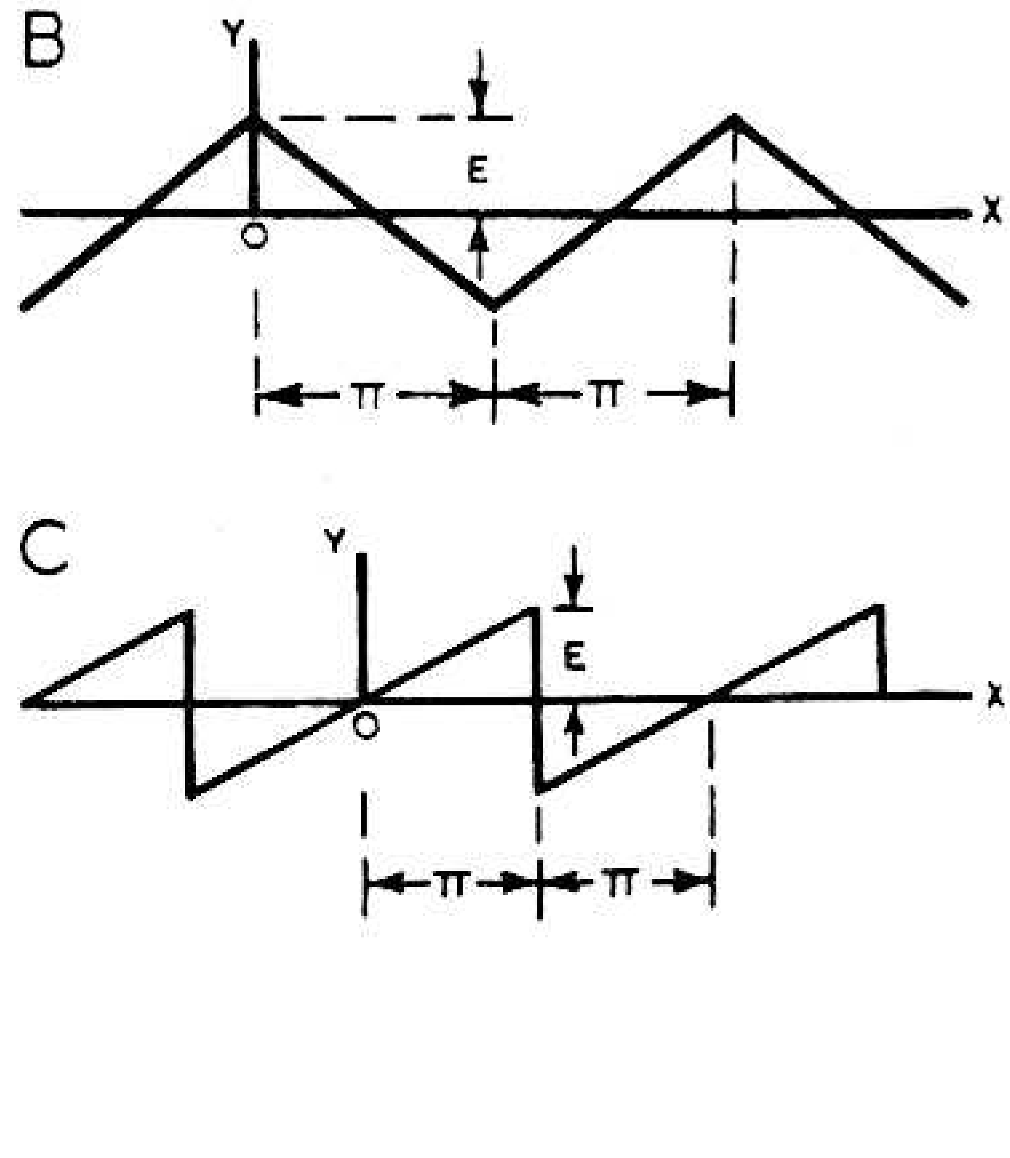
Triangular wave (B)

$$y = \frac{8E}{\pi^2}\left(\cos \theta + \frac{\cos 3\theta}{9} + \frac{\cos 5\theta}{25} + \ldots\right) \tag{10}$$

Sawtooth wave (C)

$$y = \frac{2E}{\pi}\left(\sin \theta - \frac{\sin 2\theta}{2} + \frac{\sin 3\theta}{3} - \frac{\sin 4\theta}{4} + \ldots\right) \tag{11}$$

FIG. 6.43



Short rectangular pulse (D)

$$y = E\left[k + \frac{2}{\pi}\left(\sin k\pi \cos\theta + \frac{\sin 2k\pi \cos 2\theta}{2} + \ldots + \frac{\sin nk\pi \cos n\theta}{n} + \ldots\right)\right] \tag{12}$$

Half-wave rectifier output (E)

$$y = \frac{E}{\pi}\left(1 + \frac{\pi\cos\theta}{2} + \frac{2\cos 2\theta}{3} - \frac{2\cos 4\theta}{15} + \frac{2\cos 6\theta}{35} - \ldots \ldots (-1)^{n/2+1}\frac{2}{n^2-1}\cos n\theta \ldots\right) \text{ (n even)} \tag{13}$$

Full wave rectifier output (F)

$$y = \frac{2E}{\pi}\left(1 + \frac{2\cos 2\theta}{3} - \frac{2\cos 4\theta}{15} + \frac{2\cos 6\theta}{35} - \ldots \ldots (-1)^{n/2+1}\frac{2\cos n\theta}{n^2-1}\ldots\right) \text{ (n even)} \tag{14}$$

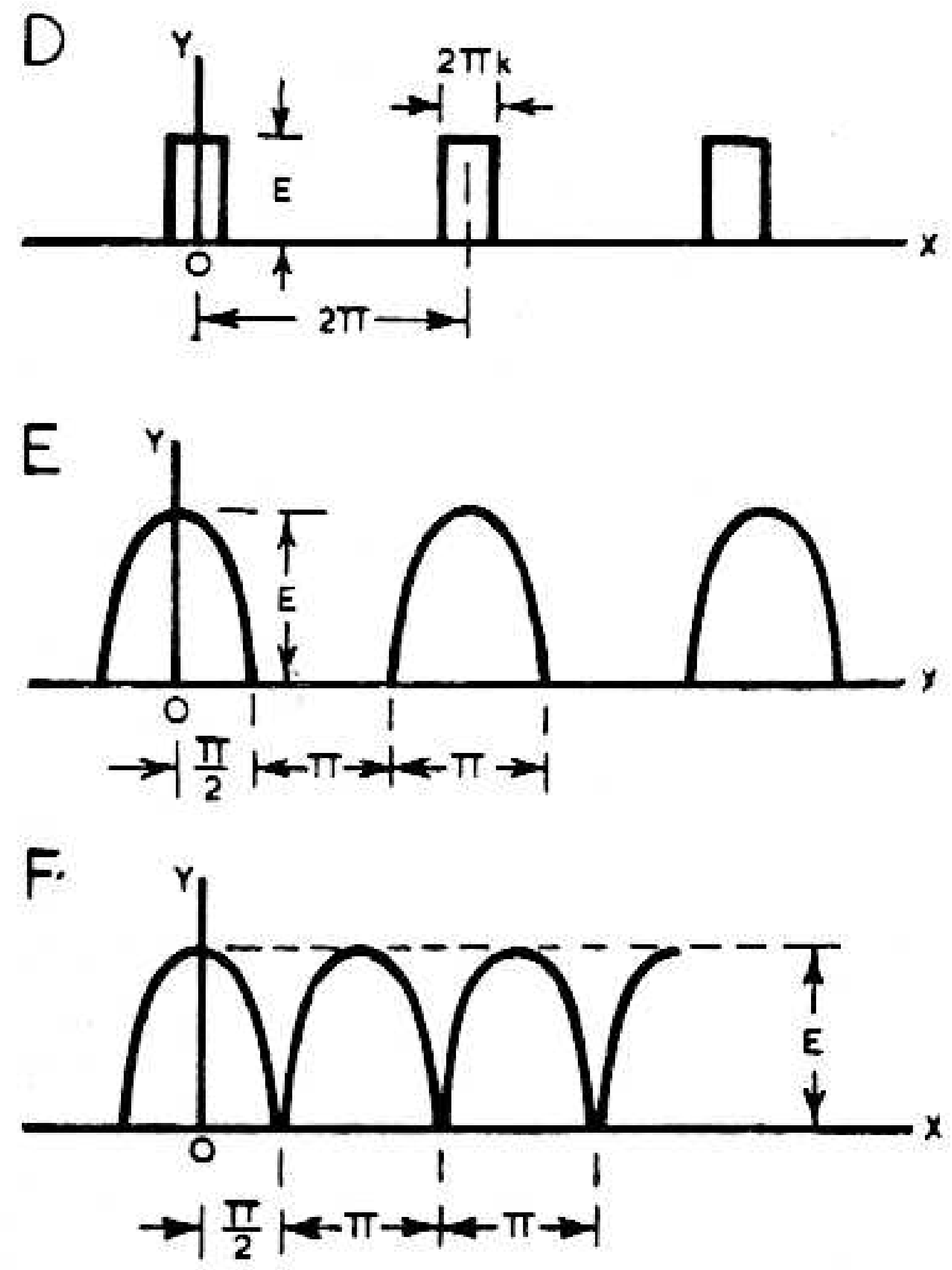


Fig. 6.43

(ii) Other applications of the Fourier Series

The Fourier Series is particularly useful in that it may be applied to functions having a finite number of discontinuities within the period, such as rectangular and saw-tooth periodic pulses.

The Fourier Series may be put into the exponential form, this being useful when the function lacks any special symmetries.

The Fourier Series may also be applied to non-periodic functions.

For information on these applications, see the list of references—Sect. 9(B).

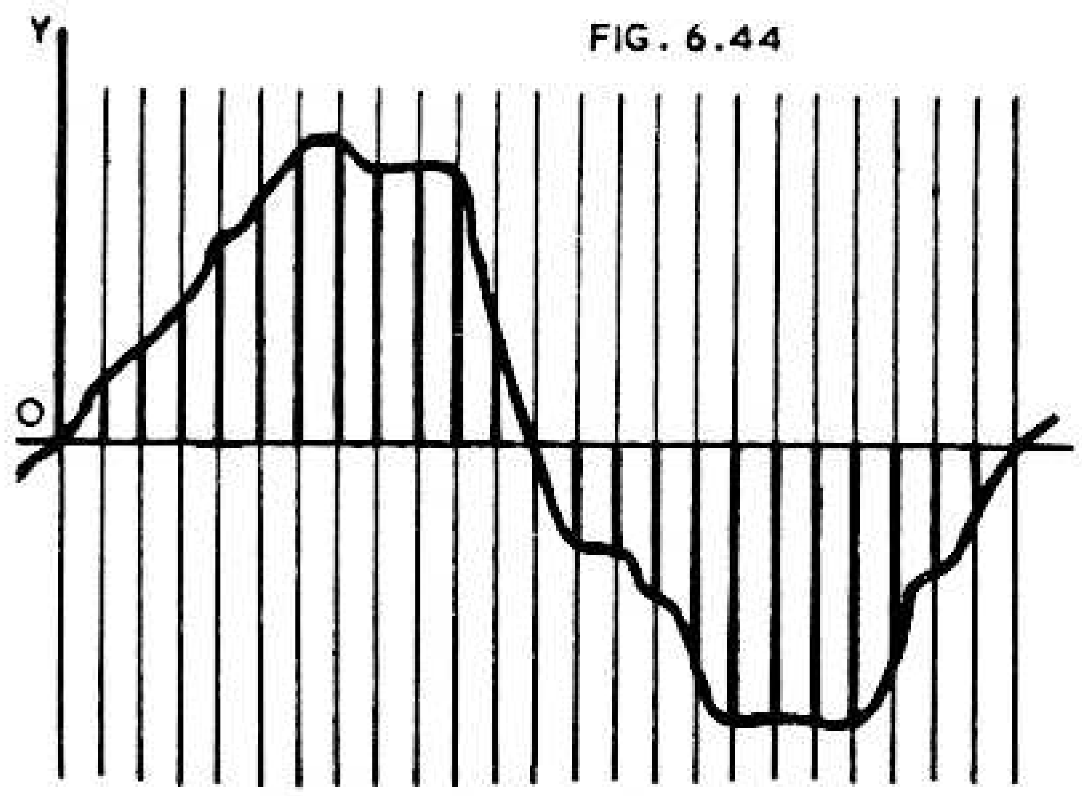
(iii) Graphical Harmonic Analysis

Any irregular waveform may be analysed to determine its harmonic content, and the general method is to divide the period along the X axis into a suitable number of divisions (e.g. Fig. 6.44 with 24 ordinates), the accuracy increasing with the number of divisions.

Ordinates are drawn at each point on the X axis and the height of each ordinate is measured. The minimum number of ordinates over the cycle must be at least twice the power of the highest harmonic which it is desired to calculate. Various methods for carrying out the calculations have been described. Some are based on equal divisions of time (or angle) while others are on equal divisions of voltage.

In the harmonic analysis of the distortion introduced by valves on resistive loads, it is possible to make use of certain properties which simplify the calculations :

FIG. 6.44



(1) All such distortion gives a waveform which is symmetrical on either side of the vertical lines (ordinates) at the positive and negative peaks.

(2) It is therefore only necessary to analyse over half the cycle, from one positive peak to the following negative peak, or *vice versa.*

(3) Even harmonic distortion results in positive and negative half cycles of different shape and area, thus causing a steady (" rectified ") component.

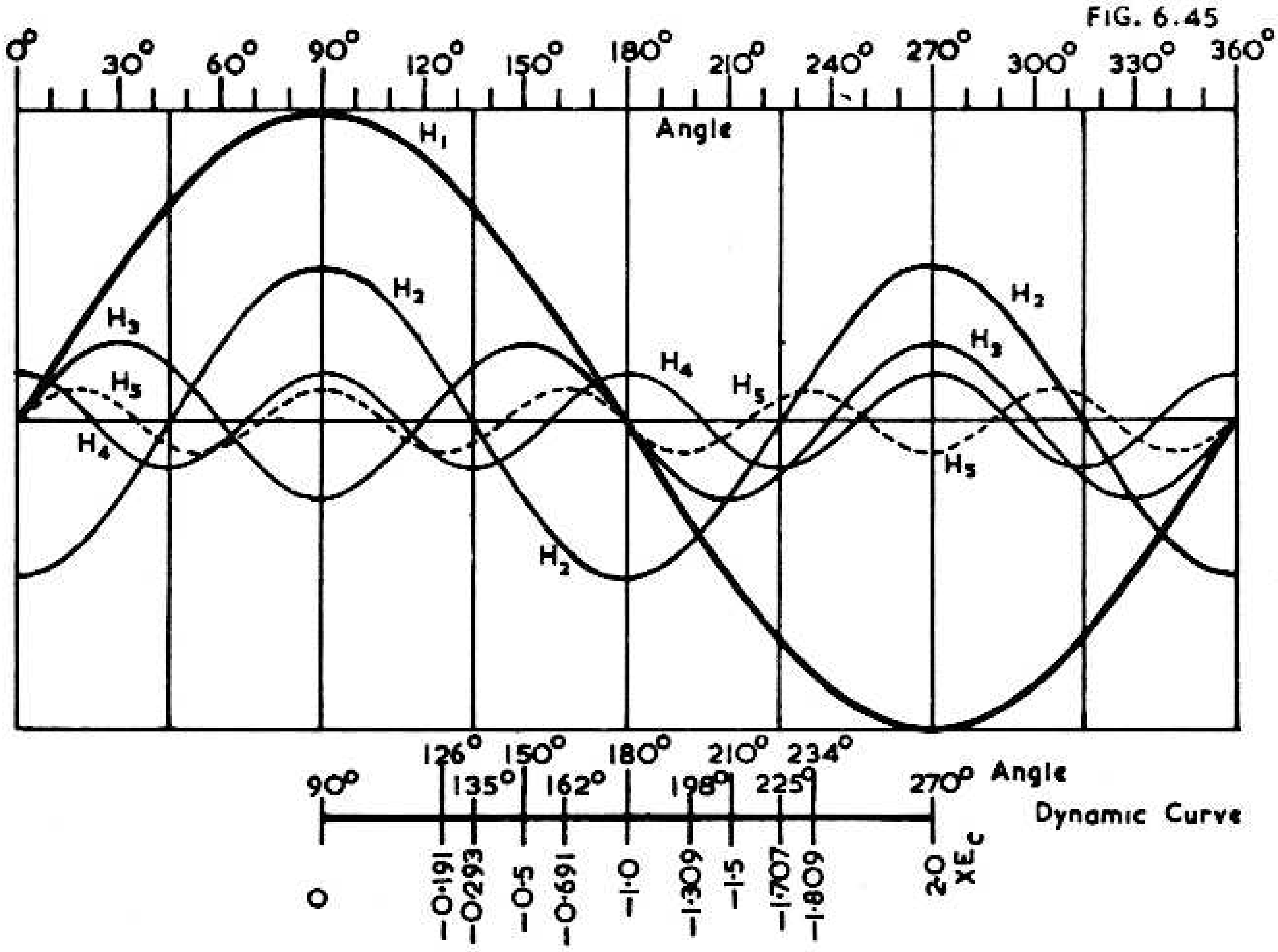
(4) Odd harmonic distortion results in distorted waveform, but with the positive and negative half cycles similar in shape.

(5) Even harmonics are in phase with the positive fundamental peak, and out of phase with the negative peak, or *vice versa* ; they are always maxima when the fundamental is zero.

(6) Odd harmonics are always exactly in phase or 180° out of phase with both positive and negative fundamental peaks, and are zero when the fundamental is zero.

The relative phases of the fundamental (H_1) and the harmonics (up to H_5) are shown in Fig. 6.45. The fundamental and third, fifth and higher order odd harmonics have zero amplitude at 0°, 180° and 360° on the fundamental scale. The second,

FIG. 6.45



fourth, and higher order even harmonics reach their maximum values (either positive or negative) at 0°, 90°, 180°, 270° and 360° on the fundamental scale.

The amplitudes of the harmonics as drawn in Fig. 6.45 have been exaggerated for convenience in drawing, while their relative magnitudes are quite arbitrary. Their relative phases are, however, quite definite.

In proceeding with Graphical Harmonic Analysis it may be shown that it is possible to select **thirteen points** on the X axis which will enable the exact values of the first, second, third and fourth harmonics to be calculated (Ref. C9) on the assumption that there are no harmonics of higher order than the fifth, or that these are negligibly small. These points are limited to the range from 90° to 270° on the fundamental scale, as in Fig. 6.45. They may be expressed in terms of the grid voltage E_c, the static operating point being $-E_c$ and the operating point swinging from $E_c = 0$ on the one side to $2E_c$ on the other side.

It is only necessary to determine the plate currents at the specified grid voltages, to insert these into the formulae given in the article and to calculate the values of the harmonics.

The preceding exact method has been approximated by R.C.A. (Ref. C10) to give greater ease in handling. In the approximation there are **eleven specified points** in place of 13 in the exact form, the values of the grid voltages being (see Fig. 6.45) : 0 ; $-0.191E_c$; $-0.293E_c$; $-0.5E_c$; $-0.691E_c$; $-E_c$; $-1.309E_c$; $-1.5E_c$; $-1.707E_c$; $-1.809E_c$; $-2.0E_c$.

These have been approximated by R.C.A. to the nearest decimal point, and the approximate values have been used in the " eleven selected ordinate method " of Chapter 13 Sect. 3(iv)D and Fig. 13.24.

The equation giving the second harmonic distortion—eqn. (28) in Chapter 13 Sect. 3(iv)—only requires the values of plate current at three points. This is an exact form and is used for triodes in Chapter 13 Sect. 2(i) eqns. (6) to (7b) inclusive and Fig. 13.2.

The " five selected ordinate method," described in Chapter 13 Sect. 3(iv)A and used for calculating second and third harmonic distortion in pentodes, is exact provided that there is no harmonic higher than the third. It is, however, a very close approximation under all normal conditions. The same remarks also apply to the **simple method for calculating third harmonic distortion** in balanced push-pull amplifiers, described in Chapter 13 Sect. 5(iii) eqn. (23) and Fig. 13.37.

An alternative method, based on **equal grid voltage divisions**, has been devised by Espley (Ref C6). This gives harmonics up to one less than the number of voltage points. Two applications are described in Chapter 13 Sect. 3(iv)—**five ordinates** giving second, third and fourth harmonic distortion, and **seven ordinates** giving up to sixth harmonic distortion.

When the loadline is a closed loop, as occurs with a partially reactive load, these conditions and equations do not apply, or are only approximated.

SECTION 9 : REFERENCES

(A) HELPFUL TEXTBOOKS ON MATHEMATICS FOR RADIO

Elementary :

Sawyer, W. W. " Mathematician's Delight " (a Pelican Book, A121, published by Penguin Books, England and U.S.A. 1943). Possibly the best introduction to mathematics.

Cooke, N. M. and J. B. Orleans, " Mathematics essential to electricity and radio " (McGraw-Hill, New York and London, 1943). Highly recommended for general use. 418 pages.

Everitt, W. L. (Editor) " Fundamentals of radio " (Prentice-Hall Inc. New York, 1943). Chapter 1 only.

" Radio Handbook Supplement " (The Incorporated Radio Society of Great Britain, London, 2nd ed., 1942). Chapters 2, 9.

" Radio Handbook " (10th edit., Editors and Engineers, Los Angeles, California). Chapter 28. (Not 11th edition.)

Basic :

Colebrook, F. M. " Basic Mathematics for Radio Students " (Iliffe and Sons Ltd., London, 1946).

Complete textbooks, commencing from elementary level :

Dull, R. W. " Mathematics for Engineers " (McGraw-Hill Book Co., New York and London, 2nd edit. 1941). Covers whole ground. 780 pages.

Cooke, N. M. " Mathematics for Electricians and Radiomen " (McGraw-Hill Book Coy., New York and London, 1942). Generally at lower level than R. W. Dull and less comprehensive. Useful for those with limited mathematical background. 604 pages.

Smith, Carl E. " Applied Mathematics for Radio and Communication Engineers " (McGraw-Hill Book Coy., New York and London, 1945). Excellent treatment in limited space. 336 pages.
Wang, T. J. " Mathematics of Radio Communications " (D. van Nostrand Coy., New York, 1943). 371 pages.
Rose, W. N. " Mathematics for Engineers " Parts 1 and 2 (Chapman and Hall Ltd., London, 2nd edition 1920).

Textbooks commencing at higher level :

Warren, A. G. " Mathematics Applied to Electrical Engineering " (Chapman and Hall, London, 1939). 384 pages.
Sokolnikoff, I. S. and E. S. " Higher Mathematics for Engineers and Physicists " (McGraw-Hill Book Coy., New York and London, 2nd edit., 1941). 587 pages.
Toft, L., and A. D. D. McKay " Practical Mathematics " (Sir Isaac Pitman and Sons Ltd., London, 2nd edit., 1942). 612 pages.
Jaeger, J. C. " An Introduction to Applied Mathematics " (Oxford University Press).

Limited application :

Sturley, K. R. " Radio Receiver Design " Part 1 (Chapman and Hall, London, 1943). Appendix 1A—j notation ; 2A—Fourier Series.
Golding, E. W., " Electrical Measurements and Measuring Instruments " (Sir Isaac Pitman and Sons Ltd., London, 3rd edit., 1944). Chapter 15—Wave forms and their determination.
Lawrence, R. W. " Principles of Alternating Currents " (McGraw-Hill Book Coy., New York and London, 2nd ed., 1935). Chapter 4—Non-sinusoidal waves.
Eshbach, O. W. " Handbook of Engineering Fundamentals " (John Wiley and Sons, New York ; Chapman and Hall, London, 1944). Section 1—Mathematical and Physical Tables : Section 2—Mathematics.
Also other handbooks.

(B) REFERENCES TO FOURIER ANALYSIS

[see also references under (C) Graphical Harmonic Analysis]

B1. Hallman, L. B. " A Fourier analysis of radio-frequency power amplifier wave forms," Proc. I.R.E. 20.10 (Oct. 1932) 1640.
B2. Lockhart, C. E. " Television waveforms—an analysis of saw-tooth and rectangular waveforms encountered in television and cathode ray tube practice, " Electronic Eng. 15.172 (June, 1942) 19.
B3. Williams, H. P. " Fourier analysis by geometrical methods," W.E. 21.246 (March, 1944) 108.
B4. de Holzer, R. C. " The harmonic analysis of distorted sine waves," Electronic Eng. 17.208 (June, 1945) 556 ; 17.209 (July, 1945) 606.
B5. Moss, H. " Complex waveforms," Electronic Eng. The harmonic synthesiser, 18.218 (April, 1946) 113 ; Analysis of complex waves, 18.220 (June, 1946) 179 ; 18.222 (Aug., 1946) 243.
B6. Furst, U. R. " Harmonic analysis of overbiased amplifiers," Elect. 17.3 (March, 1944) 143. Gives curves of harmonics of idealized straight characteristic and sharp angle cut-off.
B7. Espley, D. C., " Harmonic Analysis by the method of central differences " Phil. Mag. 28.188 (Sept., 1939) 338.

(C) REFERENCES TO GRAPHICAL HARMONIC ANALYSIS

Books

C1. Manley, R. G. " Waveform analysis " (Chapman and Hall, London, 1945) and most radio and electrical engineering text books.

Articles

Equal time divisions

C2. Kemp, P. " Harmonic analysis of waves containing odd and even harmonics," Electronic Eng. 15.172 (June, 1942) 13 (12 ordinates per cycle).
C3. Denman, R. P G. " 36 and 72 ordinate schedules for general harmonic analysis," Elect. 15.9 (Sept., 1942) 44. Correction 16.4 (April, 1943) 214.
C4. Cole, L. S. " Graphical analysis of complex waves," Elect. 18.10 (Oct., 1945) 142. (6, 8 and 12 points per cycle.)
C5. Levy, M. M. " Fourier Series," Jour. Brit. I.R.E. 6.2 (March-May, 1946) 64. (Calculations up to 160 harmonics)
and many other references.

Equal voltage divisions

C6. Espley, D. C. " The calculation of harmonic production in thermionic valves with resistive loads," Proc. I.R.E. 21.10 (Oct., 1933) 1439.

Selected ordinates

C7. Hutcheson, J. A. " Graphical Harmonic Analysis," Elect. 9.1 (Jan., 1936) 16 (odd and even harmonics in amplifiers).
C8. Chaffee, E. L. " A simplified harmonic analysis," Review of Scientific Instruments, 7 (Oct., 1936) 384 (Gives 5, 7, 9, 11 and 13 point analysis).
C9. Mouromtseff, I. E., and H. N. Kozanowski," A short-cut method for calculation of harmonic distortion in wave modulation," Proc. I.R.E. 22.9 (Sept., 1934) 1090.
C10. R.C.A. Application Note " Use of the plate family in vacuum tube power output calculations " No. 78 (July, 1937).